

# Transport and large deviations for Schrodinger operators and Mather measures

A. O. Lopes and Ph. Thieullen

**Abstract** In this mainly survey paper we consider the Lagrangian  $L(x, v) = \frac{1}{2} |v|^2 - V(x)$ , and a closed form  $w$  on the torus  $\mathbb{T}^n$ . For the associated Hamiltonian we consider the the Schrodinger operator  $\mathbf{H}_\beta = -\frac{1}{2\beta^2} \Delta + V$  where  $\beta$  is large real parameter. Moreover, for the given form  $\beta w$  we consider the associated twist operator  $\mathbf{H}_\beta^w$ . We denote by  $(\mathbf{H}_\beta^w)^*$  the corresponding backward operator. We are interested in the positive eigenfunction  $\psi_\beta$  associated to the the eigenvalue  $E_\beta$  for the operator  $\mathbf{H}_\beta^w$ . We denote  $\psi_\beta^*$  the positive eigenfunction associated to the the eigenvalue  $E_\beta$  for the operator  $(\mathbf{H}_\beta^w)^*$ . Finally, we analyze the asymptotic limit of the probability  $\nu_\beta = \psi_\beta \psi_\beta^*$  on the torus when  $\beta \rightarrow \infty$ . The limit probability is a Mather measure. We consider Large deviations properties and we derive a result on Transport Theory. We denote  $L^-(x, v) = \frac{1}{2} |v|^2 - V(x) - w_x(v)$  and  $L^+(x, v) = \frac{1}{2} |v|^2 - V(x) + w_x(v)$ . We are interest in the transport problem from  $\mu_-$  (the Mather measure for  $L^-$ ) to  $\mu_+$  (the Mather measure for  $L^+$ ) for some natural cost function. In the case the maximizing probability is unique we use a Large Deviation Principle due to N. Anantharaman in order to show that the conjugated sub-solutions  $u$  and  $u^*$  define an admissible pair which is optimal for the dual Kantorovich problem.

## 1 Introduction and basic definitions

Given a closed form  $w$  on the torus  $\mathbb{T}^n$  we consider the Lagrangian  $L(x, v) = \frac{1}{2} |v|^2 - V(x) + w$ , where  $L : T\mathbb{T}^n \rightarrow \mathbb{R}$  and  $T\mathbb{T}^n$  is the tangent bundle.

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The infimum of  $\int L(x, v) d\mu(x, v)$  among the invariant probabilities for the Euler Lagrange flow on the tangent bundle  $T\mathbb{T}^n$  is called the critical value of  $L$ . A probability which attains such infimum is called a Mather measure (see [CI] for references and general results).

We denote by  $H(x, p) = \frac{1}{2}|p|^2 + V(x)$  the associated Hamiltonian for the Lagrangian  $L(x, v) = \frac{1}{2}|v|^2 - V(x)$  and for each  $\beta \in \mathbb{R}$  we consider the corresponding Schrodinger operator  $\mathbf{H}_\beta = -\frac{1}{2\beta^2}\Delta + V$  for such Hamiltonian.

For each  $\beta$  we consider a certain associated quantum state and quantum probability on  $\mathcal{L}^2(\mathbb{T}^n)$  (associated to an eigenvalue of  $\mathbf{H}_\beta$ ) and we are interested in the limit of such probability when  $\beta \rightarrow \infty$ .

We call  $\beta$  the semiclassical parameter. In an alternative form we can take  $\hbar = \frac{1}{\beta}$  and consider the limit when  $\hbar \rightarrow 0$ .

An interesting relation of such limit probabilities with Mather measures was investigated by N. Anantharaman (see [A3] [A1] and [A2])

We will present here some of these results which are related to transport and large deviation properties.

Consider  $w(v) = \langle P, v \rangle$  a closed form  $w$  in the torus  $\mathbb{T}^n$ , where  $P$  is a vector in  $\mathbb{R}^n$ .

Suppose that  $\mu_+$  and  $\mu_-$  are respectively the Mather measures for the Lagrangians

$$L^+(x, v) = \frac{1}{2}|v|^2 - V(x) + w(v) \text{ and } L^-(x, v) = \frac{1}{2}|v|^2 - V(x) - w(v),$$

$x \in \mathbb{T}^n$  and  $V : \mathbb{T}^n \rightarrow \mathbb{R}$  smooth.

We assume the Mather measure is unique in each problem (see [CI], [Fathi1], [Fathi]).

We will follow closely the notation of the nice exposition [A1] (see also [A2] [A3]). The results presented here in the future sections are inspired in [LOP]. The main tool is the involution kernel introduced in [BLT] (see also [LM] [LOS] [LO] [LMMS] [CLO] [GLM] [LR])

We will consider the Lax-Oleinik operator  $T_t^-, t \geq 0$ , given by

$$T_t^- u_1(x) = \inf_{\gamma(t)=x, \gamma: [0, t] \rightarrow \mathbb{T}^n} \{u_1(\gamma(0)) + \int_0^t L^-(\gamma(s), \gamma'(s)) ds\}.$$

Denote by  $u$  (Lipchitz),  $u : \mathbb{T}^n \rightarrow \mathbb{R}$ , the unique (up to additive constant because  $\mu_-$  is unique) solution of  $T_t^- u = u + tE$ , for all  $t \geq 0$ , and where  $E$  is constant.

Consider the Lax-Oleinik operator  $T_t^+, t \geq 0$ , given by

$$T_t^+ u_2(x) = \inf_{\gamma(0)=x, \gamma: [0, t] \rightarrow \mathbb{T}^n} \{u_2(\gamma(t)) - \int_0^t L^-(\gamma(s), \gamma'(s)) ds\}.$$

Denote by  $u^*$  the Lipchitz function  $u^* : \mathbb{T}^n \rightarrow \mathbb{R}$ , such that,  $T_t^+(-u^*) = -u^* + Et$ . We assume that  $u$  and  $u^*$  are such  $u + u^*$  is zero in the support of  $\mu_-$ .

The function  $W(x, y)$  (which could be called the convolution kernel) is given by the bellow expression

$$- \inf_{\alpha \in C^1([0,1], \mathbb{T}^n), \alpha(0)=x, \alpha(1)=y} \left\{ \int_0^1 [-V(\alpha(s) + w(\alpha'(s)))] ds + \int_0^1 \frac{1}{2} \|\alpha'(t)\|^2 \right\}.$$

We denote by  $h(y, y)$  the Peierls barrier for the Lagrangian  $L^-$ . In the present case  $h(y, y) = u(y) + u^*(y)$ .

Main references on Transport Theory are [Vi1] [Vi2] and [Ra].

We denote by  $\mathcal{K}(\mu_+, \mu_-)$  the set of probabilities  $\hat{\mu}$  on  $\mathbb{T}^n \times \mathbb{T}^n$ , such that respectively  $\mu_+ = \pi_1^\#(\hat{\mu})$  and  $\mu_- = \pi_2^\#(\hat{\mu})$ .

Give  $c(x, y)$  we say that  $f$  and  $g$  are  $c$ -admissible if, for any  $x, y \in \mathbb{R}^n$ , we have  $f(x) - g(y) \leq c(x, y)$ . We denote by  $\mathcal{F}$  the set of such pairs  $(f, g)$ .

We will consider, for the cost function  $c(x, y) = -W(y, x)$ , a  $c$ -Kantorovich problem

$$\inf_{\hat{\mu} \in \mathcal{K}(\mu_+, \mu_-)} \int \int c(x, y) d\hat{\mu}(x, y).$$

We denote the minimizing probability by  $\hat{\mu}_{\min}$ . Note that this probability projects on the second variable on  $\mu_-$ .

Note that the transport optimal probability for  $-W$  and for  $-W + I$  (where  $I$  is the Peierl's barrier) are the same.

We point out that the projected Mather measures  $\mu^+$  and  $\mu^-$  are the same in the present case.

We will show here that the dual problem for  $-W$

$$\begin{aligned} \max \left\{ \int f(x) d\mu_+(x) - \int g(y) d\mu_-(y) \mid f(x) - g(y) \leq c(x, y) \right\} = \\ \max \left\{ \int f(x) d\mu_+(x) - \int g(y) d\mu_-(y) \mid (f, g) \in \mathcal{F} \right\}, \end{aligned}$$

has a pair of optimal solutions  $(u, u^*)$  which are the viscosity solutions of the Hamilton-Jacobi equations (fixed points of the corresponding Lax-Oleinik operators as defined above)

We can consider alternatively (the same problem)

$$\inf_{\hat{\mu} \in \mathcal{K}(\mu_+, \mu_-)} \int \int \tilde{c}(x, y) d\hat{\mu}(x, y),$$

where  $\tilde{c}(x, y) = -W(y, x) + h(y, y)$ . The introduction of a function on the variable  $y$  which vanishes in the support of  $\mu_-$  does not change the minimizing measure. However, this new problem have a different optimal pair.

We denote by  $\mathcal{W}_x^h$  the Brownian motion in  $\mathbb{R}^n$  (with coefficient  $h$ , that is, at time  $t = 1$  the variance is  $\sqrt{h}$ ) beginning at  $x$ , and  $\mathcal{W}_{x,y,t}^h$  its disintegration at the point  $y$  and at the time  $t$ .

Consider the Schrodinger  $\mathbf{H}^h = -\frac{h^2}{2}\Delta + V$  (where  $V$  is the periodic extension to  $\mathbb{R}^n$ ) which acts on real (periodic) functions defined in  $\mathbb{R}^n$ . It is known that  $\mathbf{H}$  has pure point spectrum (see [Dav] and [LS]).

Note that

$$-\frac{1}{h}\mathbf{H}^h = \frac{h}{2}\Delta - \frac{1}{h}V.$$

The Kernel  $K(x,y,t)$  of the extension of  $e^{-\frac{t}{h}H}$  to an integral operator is (see [A1])

$$K(x,y,t) = \int e^{-\frac{1}{h}\int_0^t V(\alpha(s))ds} \mathcal{W}_{x,y,t}^h(d\alpha).$$

Given

$$L^w(x,v) = \frac{1}{2}v^2 - V(x) - w(v) = \frac{1}{2}v^2 - V(x) - \langle P, v \rangle,$$

the corresponding Hamiltonian  $H^w(x,p)$  via Legendre transform is

$$H^w(x,p) = \frac{\|p + P\|^2}{2} + V(x).$$

In the same way, for

$$L^+(x,v) = \frac{1}{2}v^2 - V(x) + w(v) = \frac{1}{2}v^2 - V(x) - \langle P, v \rangle,$$

the corresponding Hamiltonian  $H^{w^*}(x,p)$  is

$$H^{w^*}(x,p) = \frac{\|p - P\|^2}{2} + V(x).$$

Consider, a certain point  $x_0 = \mathcal{O} \in \mathbb{R}^n$  fixed (on the universal cover of the torus). As the form  $w$  on the torus is closed, it is exact on the lifting to the universal cover, then, the value  $\int_{x_0}^x w$  does not depend on the path we choose to connect  $x_0$  to  $x$ .

## 2 Transport in the configuration space for the Aubry-Mather problem

For each real value  $\beta$  we consider the operator

$$\mathbf{H}_\beta^w = e^{-\beta \int_{x_0}^x w} \circ \mathbf{H}_\beta \circ e^{\beta \int_{x_0}^x w} = e^{-\beta \int_{x_0}^x w} \circ \left(-\frac{1}{2\beta^2}\Delta + V\right) \circ e^{\beta \int_{x_0}^x w}.$$

We can consider such operator acting on the torus or on  $\mathbb{R}^n$ . When we consider the Brownian motion we should consider, off course, its action on  $\mathbb{R}^n$ .

The Kernel  $K(x,y,t)$  of the extension of  $e^{t\beta\mathbf{H}_\beta^w}$  to an integral operator is given by

$$K_\beta(x, y, t) = \int e^{-\beta \int_0^t V(\alpha(s)) ds + \beta \langle P, (y-x) \rangle} \mathcal{W}_{x,y,t}^{\beta^{-1}}(d\alpha).$$

Note that above we consider the integral

$$\beta \int_0^t [-V(\alpha(s)) + w_{\alpha(s)}(\alpha'(s))] ds.$$

$\mathbf{H}_\beta^w$  is not self adjoint but has a real pure point spectrum.

We denote by  $E_\beta$  the maximum eigenvalue of  $\mathbf{H}_\beta^w$  (acting on real functions) and  $\psi_\beta$  is the corresponding normalized real eigenfunction in  $\mathcal{L}^2(\mathbb{T}^n, dx)$ . The positive eigenfunction  $\psi_\beta$  is unique if we assume its norm is 1. It's the only totally positive eigenfunction of  $\mathbf{H}_\beta^w$  (see [A1] expression (3.15)). The eigenvalue is simple and isolated (see appendix on [A2]).

For each real value  $\beta$  we consider the  $w$ -backward operator

$$\mathbf{H}_\beta^{w*} = e^{\beta \int_{x_0}^x w} \circ \mathbf{H}_\beta \circ e^{-\beta \int_{x_0}^x w} = e^{\beta \int_{x_0}^x w} \circ \left( -\frac{1}{2\beta^2} \Delta + V \right) \circ e^{-\beta \int_{x_0}^x w}.$$

We will be interested in high values of  $\beta$ .

$E_\beta$  is the maximum eigenvalue of  $\mathbf{H}_\beta^{w*}$  and we denote by  $\psi_\beta^*$  the corresponding real eigenfunction in  $\mathcal{L}^2(\mathbb{T}^n, dx)$ . Similar properties to the case of  $\psi_\beta$  are true for such eigenfunction. We assume  $\int \psi_\beta^*(x) dx = 1$  and also  $\int \psi_\beta(x) \psi_\beta^*(x) dx = 1$ .

$\mathbf{H}_\beta^{w*} \circ \mathbf{H}_\beta^w$  is self adjoint.

We will be interested here in the probabilities

$$\nu_\beta(dx) = \psi_\beta(x) \psi_\beta^*(x) dx.$$

The probability  $\nu_\beta(dx) = \psi_\beta(x) \psi_\beta^*(x) dx$  is stationary for the Markov operator

$$\mathcal{Q}^t(f)(x) = e^{-tE_w} \psi_\beta(x)^{-1} e^{t\mathbf{H}_\beta^w} (\psi_\beta f)(x)$$

on the torus  $\mathbb{T}^n$  (see [A2]).

The correct point of view is to consider  $\psi_\beta$  as an eigenfunction and  $\rho_\beta = \psi_\beta^*(x) dx$  as an eigen-probability for the semi-group  $t \rightarrow e^{t\mathbf{H}_\beta^w}$ .

Consider

$$u_\beta = -\frac{\log \psi_\beta}{\beta} \quad \text{and} \quad u_\beta^* = -\frac{\log \psi_\beta^*}{\beta}.$$

It is known that the following equalities are true:

$$-\frac{1}{2\beta} \Delta u_\beta + H^w(x, d_x u_\beta) = E_\beta,$$

and

$$-\frac{1}{2\beta} \Delta u_\beta^* + H^w(x, -d_x u_\beta^*) = E_\beta,$$

The  $\beta$ -families of functions  $u_\beta$  and  $u_\beta^*$  are equi-Lipschitzians and we can obtain from this fact convergent subsequences. We assume here the Mather measure is unique, and therefore the limits exist in the uniform convergence topology, that is

$$\lim_{\beta \rightarrow \infty} u_\beta = u \text{ and } \lim_{\beta \rightarrow \infty} u_\beta^* = u^*.$$

It is known that  $\lim_{\beta \rightarrow \infty} E_\beta$  exist and we denote this value by  $E$ .

By stability of the viscosity solutions, the limits  $u$  and  $u^*$  are, respectively, viscosity solutions of the equations

$$H^w(x, d_x u) = E \text{ and } H^w(x, -d_x u^*) = E$$

We assume also that the Mather measure  $\mu$  for the lagrangian  $L^w$  is unique. In this case it is known (see for instance [A1] [A2]) that in the weak topology

$$\lim_{\beta \rightarrow \infty} \nu_\beta = \mu.$$

In proposition 3.11 in [A1] the following Large Deviation Principle is obtained (see also [A2] [A3]):

**Proposition 1.** *Suppose the Mather measure is unique. Suppose also that in the uniform convergence topology*

$$\lim_{\beta \rightarrow \infty} u_\beta = u \text{ and } \lim_{\beta \rightarrow \infty} u_\beta^* = u^*.$$

*Then, for  $I(x) = u(x) + u^*(x)$  (from the normalization we choose before  $I(x) \geq 0$ ), we have*

*1) for any open set  $O \subset \mathbb{T}^n$ ,*

$$\liminf_{\beta \rightarrow \infty} \frac{1}{\beta} \log \nu_\beta(O) = - \inf_{x \in O} \{I(x)\},$$

*and,*

*2) for any closed set  $F \subset \mathbb{T}^n$ ,*

$$\limsup_{\beta \rightarrow \infty} \frac{1}{\beta} \log \nu_\beta(F) = - \inf_{x \in F} \{I(x)\}.$$

It follows from Varadhan's Integral Lemma (section 4.3 in [DZ]) that, for any  $C^\infty$  function  $F(x)$ ,

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \int e^{\beta F(x)} d\nu_\beta(x) = \sup_{x \in \mathbb{T}^n} \{F(x) - I(x)\}.$$

The  $W_\beta^t$ -Kernel is defined by

$$e^{W_\beta^t(y,x)} = \int e^{-\beta \int_0^t V(\alpha(s)) ds - \beta \int_y^x w \mathscr{W}_{y,x,t}^{\beta^{-1}}(d\alpha)} =$$

$$\int e^{-\beta \int_0^t V(\alpha(s)) ds - \beta \langle P, (x-y) \rangle} \mathcal{W}_{y,x,t}^{\beta^{-1}}(d\alpha).$$

Note the plus sign on  $V$ .

Note that we exchange  $x$  and  $y$  above (with respect to the previous considerations).

It is known (see [A1]) that for any  $\beta$  and any  $t$

$$\psi_\beta(x) = \int e^{W_\beta^t(y,x)} \psi_\beta^*(y) dy = \int e^{W_\beta^t(y,x)} \frac{1}{\psi_\beta(y)} d\nu_\beta(y)$$

Now from Schilder's Theorem and Varadhan's Integral Lema (see [DZ] also Theorem 4.3.9 in [A2])

$$-W(y,x) := -\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log e^{W_\beta^{\frac{1}{\beta}}(y,x)} =$$

$$\inf_{\alpha \in C^1([0,1], \mathbb{T}^n), \alpha(0)=y, \alpha(1)=x} \left\{ \int_0^1 [-V(\alpha(s) + w_{\alpha(s)}(\alpha'(s))] ds + \int_0^1 \frac{1}{2} \|\alpha'(t)\|^2 \right\}$$

Note above the plus sign on  $w$ .

The function  $W(y,x)$  is the function  $-I(y,x)$  in the notation of [A2].

For any  $\beta$

$$\frac{1}{\beta} \log(\psi_\beta(x)) = \frac{1}{\beta} \log \left( \int e^{W_\beta^{\frac{1}{\beta}}(y,x)} \psi_\beta(y)^{-1} d\nu_\beta(y) \right).$$

Taking limits as  $\beta \rightarrow \infty$  and using the Varadhan's integral Lemma once more we get

$$-u(x) = \sup_{z \in \mathbb{T}^n} \{W(z,x) + u(z) - I(z)\}.$$

Therefore, for any  $x,y$  we get that

$$-u(y) - u(x) \geq W(y,x) - I(y).$$

From this we get

**Proposition 2.**

$$u(y) + u(x) \leq -W(y,x) + I(y) = c(y,x).$$

and the pair  $(u, u)$  is  $(-W+I)$ -admissible.

In the same way the pair  $(u, u^*)$  is  $-W$ -admissible.

**Proposition 3.** *If  $\hat{\eta}$  is an optimal minimizing transport probability for  $c$  and if  $(f, f^\#)$  is an optimal pair in  $\mathcal{F}$ , then the support of  $\hat{\eta}$  is contained in the set*

$$\{(x,y) \in M \times M \text{ such that } (f(x) - f^\#(y)) = c(x,y)\}.$$

**Proof:** It follows from the primal and dual linear programming problem formulation. The condition above is called the complementary slackness condition (see [EG]).  $\square$

If one finds  $\hat{\eta}$  an admissible pair  $(f, f^\#)$  satisfying the above claim (for the support) one solves the Kantorovich problem, that is, one finds the optimal transport probability  $\hat{\eta}$ .

From the above it follows.

**Proposition 4. Proposition:** *For  $(x, y)$  in the support of  $\hat{\mu}$  we have*

$$u(x) + u(y) = -W(y, x) + h(y, y) = c(x, y),$$

or

$$u(x) + u(y) = -W(y, x) + (u^*(y) + u(y)).$$

*This means, for  $(x, y)$  in the support of  $\hat{\mu}$*

$$u(x) - u^*(y) = -W(y, x).$$

In other words, for any  $x, y$  in the support of  $\hat{\mu}_{min}$  we have that  $u(x)$  is given by

$$\alpha \in C^1([0, 1], \mathbb{T}^n), \alpha(0)=y, \alpha(1)=x \quad \left\{ \int_0^1 [-V(\alpha) + w(\alpha')] ds + \int_0^1 \frac{1}{2} \|\alpha'\|^2 ds \right\} + u^*(y).$$

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