Stochastics and Dynamics

CONVERGENCE IN DISTRIBUTION OF THE PERIODOGRAM OF CHAOTIC PROCESSES

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In this work we analyze the convergence in distribution sense of the periodogram function (to the spectral density function) based on a time series of a stationary process $X_t = (\varphi \circ T^t)(X_0)$ obtained from the iterations of a continuous transformation T invariant for an ergodic probability μ and a continuous function φ taking values in $\mathbb{R}$. We only assume a certain rate of convergence to zero for the autocovariance coefficient of the stochastic process, that is, we assume there exist $C > 0$ and $\beta > 2$ such that $|\gamma_X(h)| \leq C|h|^{-\beta}$, for all $h \in \mathbb{N}$, where $\gamma_X(h) = \int (\varphi \circ T^h)(x)\, \varphi(x) d\mu(x) - (\int \varphi(x) d\mu(x))^2$ is the h-autocovariance of the process. Our result applies to the case of exponential decay of correlation (or covariance), as it happens for a continuous expanding transformation T on the circle and a Holder potential φ. It can also be applied to the case when the transformation T has a fixed point with derivative equal to one.

Keywords: Periodogram Function, Convergence in Distribution, Chaotic dynamics, Stochastic Processes.

1. Introduction

Here we consider the stochastic process $\{X_t\}_{t\in\mathbb{N}}$ obtained from the iterations of a continuous transformation T (not necessarily invertible) from the unit interval (or the circle) to itself and μ an ergodic probability invariant under T. This stochastic and stationary process $\{X_t\}_{t\in\mathbb{N}}$ is given by

$$X_t \equiv (\varphi \circ T^t)(X_0) = \varphi(T^t(X_0)) = (\varphi \circ T)(X_{t-1}), \text{ for } t \in \mathbb{N}, \tag{1.1}$$

where φ is a continuous map $\varphi : [0,1) \to \mathbb{R}$ and X_0 is distributed over $[0,1)$ according to μ.

Considering the identification of the circle $z \in S$ with $x \in [0,1)$ by $z \equiv e^{2\pi x i}$, from now on we can use either one of the two forms $T : S \to S$ or $T : [0,1) \to [0,1)$.

We consider μ a probability absolutely continuous with respect to the Lebesgue measure on the unit interval with density ϕ, that is, $d\mu(x) = \phi(x)dx$.

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We shall assume a certain rate of convergence to zero for the autocovariance function of such stochastic processes. We will denote by $\gamma_X(\cdot)$ the autocovariance function of the process $\{X_t\}_{t\in\mathbb{N}}$, that is,

$$\gamma_X(h) \equiv \mathbb{E}_\mu(X_h X_0) - \mathbb{E}_\mu(X_0)^2, \text{ for } h \in \mathbb{N}.$$

The *spectral density function* of the process $\{X_t\}_{t\in\mathbb{N}}$ defined in the expression (1.1) is given by

$$f_X(\lambda) = \frac{1}{2\pi}\sum_{h\in\mathbb{N}} \gamma_X(h)e^{-i\lambda h}, \text{ for } \lambda \in (0, 2\pi], \tag{1.2}$$

where $\gamma_X(\cdot)$ is the autocovariance function of the process.

In this work we analyze the convergence in distribution sense, to the spectral density function $f_X(\cdot)$, of the periodogram function associated to a time series $T^h(x_0), 1 \leq h \leq N$, obtained from a x_0 chosen with probability one according to the measure μ. This periodogram function is given by

$$I(\lambda_k) = f_N(\lambda_k)\overline{f_N(\lambda_k)},$$

where

$$f_N(\lambda) = \frac{1}{2\pi\sqrt{N}}\sum_{t=1}^{N} \varphi(T^t(x_0))e^{-i\lambda t}, \quad \lambda \in (0, 2\pi], \tag{1.3}$$

$\overline{f_N(\cdot)}$ indicates the complex conjugate of $f_N(\cdot)$ and

$$\lambda_k = \frac{2\pi k}{N}, \text{ for } k = 0, 1, \cdots, N,$$

is the k-th discrete Fourier frequency.

Note that the periodogram function depends on x_0 and N (large). Our main theorem shows a mathematical proof that one obtains an approximation of the spectral density function $f_X(\cdot)$ by means of the periodogram (see [2]). See also remark 2.1.

We assume, without loss of generality, that

$$\mathbb{E}_\mu(X_0) = \int \varphi(x)d\mu(x) = \int \varphi(x)\phi(x)dx = 0.$$

If we consider a non-invertible transformation T, for the negative integers h, the values $\gamma_X(h)$ do not make sense, but there is a general procedure to solve this problem going to the natural extension of T (see [6]). We assume here that there exists such natural extension and we refer the reader to section 5.3 in [7] for the procedure of extending $\gamma_X(h)$ for negative values of h. In [1] several examples of such general procedure are also presented.

Another alternative is not consider the natural extension and then take an expression like $\frac{1}{\pi}\sum_{h\in\mathbb{N}}\gamma_X(h)\cos(\lambda h)$ instead of $\frac{1}{2\pi}\sum_{h\in\mathbb{N}}\gamma_X(h)e^{-i\lambda h}$ in the definition of the spectral density function. We prefer to use here the expression with terms of the form $e^{-ih\lambda}$.

Assumption. *We will assume that T is such that for the given $\varphi : [0,1) \to \mathbb{R}$ continuous map there exist $C > 0$ and $\beta > 2$ such that the autocovariance function of the process $\{X_t\}_{t\in\mathbb{N}}$ has the rate of convergence to zero given by*

$$|\gamma_X(h)| \leq C|h|^{-\beta}, \text{ for all } h \in \mathbb{N}. \tag{1.4}$$

This paper proceeds in the following way: Section 1 presents some general definitions related to the chaotic process of the form (1.1) while Section 2 presents the mathematical proof of the convergence in distribution sense of the periodogram function $I(\cdot)$, under our main *assumption*, given by the expression (1.4), imposed on the autocovariance function $\gamma_X(\cdot)$.

Considering $X_t = (\varphi \circ T^t)(X_0)$ as above, where X_0 is distributed over $[0,1)$ according to the ergodic probability μ, then from Birkhoff's theorem let $x_0 \in [0,1)$ be a fixed number chosen according to μ such that for all continuous function g (or indicator functions of the interval)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} g(T^j(x_0)) = \int g(x) d\mu(x).$$

We want to prove the convergence in distribution of the periodogram (to the function $f_X(\cdot)$) based on a time series $\{X_t\}_{t=1}^N$ of the process $\{X_t\}_{t\in\mathbb{N}}$ given by (1.1) beginning with μ-almost everywhere x_0. The theorem below describes this property in a precise way.

In the sequel, δ_y denotes the Dirac delta measure in y.

Theorem 1.1. *Let $\{X_t\}_{t\in\mathbb{N}}$ be the stationary zero mean process given by $X_t = (\varphi \circ T^t)(X_0)$. Let x_0 be in a set of μ-probability one and let $\gamma_X(\cdot)$ be the autocovariance function of the process $\{X_t\}_{t\in\mathbb{N}}$ such that the assumption given by (1.4) holds. Let $f_X(\cdot)$ be the spectral density function of the process $\{X_t\}_{t\in\mathbb{N}}$. Then, in the distribution sense*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k} = f_X(\lambda), \text{ for } \lambda \in (0, 2\pi],$$

where $I(\cdot)$ is the periodogram function defined by (1.3),

$$\lambda_k = \frac{2\pi k}{N}, \text{ for } k \in \{0, 1, \cdots, N\},$$

is the k-th discrete Fourier frequency and δ_{λ_k} is the Dirac delta measure with mass at λ_k.

Our main assumption includes the case where $\gamma_X(\cdot)$ exponentially decays to zero, that is, when there exists $0 < \lambda < 1$ such that

$$|\gamma_X(h)| = |\mathbb{E}_\mu(X_h X_0)| \leq C_1 \lambda^{|h|}, \text{ for } h \in \mathbb{N}, \tag{1.5}$$

where C_1 is a positive constant and $\mathbb{E}_\mu(X_t) = \mathbb{E}_\mu[(\varphi \circ T^t)(X_0)] = 0$.

In general, a transformation T of the circle with an indifferent fixed point (see [3],[4],[9] and [12]) defines a stochastic process with polynomial (not exponential) decay of correlation and the results presented here can also be applied. The $\beta > 2$ consider here is equivalent to $\gamma > 4$ in the notation of [3].

We observe that we are not considering *general fractionally integrated* processes (see [11]). In [10] several methods for estimating the natural parameter of Manneville-Pomeau processes with short and long dependence are analyzed. In this case the periodogram method is a good way to find out the velocity of decay of the autocorrelation function of the process.

The *periodogram function* is an unbiased estimator for the spectral density function $f_X(\cdot)$, even though it is not consistent (see [2]).

In applications, it is very useful to have an approximation, as close as one wants, of the function $f_X(\cdot)$ by means of the periodogram function which is in general easy to obtain.

We refer the reader to Figures 4 and 5 in [8] for a good geometrical description of the above property.

2. Convergence in Distribution of the Periodogram

In this section we will show the convergence in distribution of the periodogram function. First we need Lemma 2.1.

Lemma 2.1. *Let $f_X(\cdot)$ be the spectral density function of the process $\{X_t\}_{t\in\mathbb{N}}$ defined by the expression (1.1). Let $f_{X,r}(\cdot)$ be the truncated r-spectral density function of the process $\{X_t\}_{t\in\mathbb{N}}$ given by*

$$f_{X,r}(\lambda) = \frac{1}{2\pi} \sum_{|h|\leq r} \gamma_X(h) e^{-i\lambda h}, \ \text{for } \lambda \in (0, 2\pi], \tag{2.1}$$

for $r \in \mathbb{N}$. Then, for all $\epsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $K_0 \geq r_0$,

$$||f_X(\lambda) - f_{X,K_0}(\lambda)||_\infty < \epsilon, \ \text{for all } \lambda \in (0, 2\pi],$$

where $||g||_\infty$ means the infinity norm of the function g.

Proof: Let ϵ be a positive fixed constant. For all $K_0 \in N$, let us consider the spectral density function for all $h \in \mathbb{N}$ such that $|h| \leq K_0$, that is, the function

$f_{X,K_0}(\cdot)$ given by the expression (2.1). Then,

$$\begin{aligned} f_X(\lambda) &= \frac{1}{2\pi}\sum_{h\in\mathbb{N}} \mathbb{E}_\mu(X_h X_0)e^{-i\lambda h} \\ &= \frac{1}{2\pi}\sum_{h\in\mathbb{N}} \gamma_X(h)e^{-i\lambda h} \\ &= \frac{1}{2\pi}\sum_{|h|\leq K_0} \gamma_X(h)e^{-i\lambda h} + \frac{1}{2\pi}\sum_{|h|>K_0} \gamma_X(h)e^{-i\lambda h} \\ &= f_{X,K_0}(\lambda) + \frac{1}{2\pi}\sum_{|h|>K_0} \gamma_X(h)e^{-i\lambda h}. \end{aligned} \tag{2.2}$$

Now, since there exist $C > 0$ and $\beta > 2$ such that, for all $h \in \mathbb{N}$, we have $|\gamma_X(h)| \leq C|h|^{-\beta}$ then $\sum_{h\in\mathbb{N}} |\gamma_X(h)|$ converges.

The last term in the above expression goes to zero when r goes to infinity since $\frac{1}{2\pi}\sum_{h\in\mathbb{N}} \gamma_X(h)e^{-i\lambda h}$ converges. Therefore, given $\epsilon > 0$, there exists such r_0.

Therefore, the Lemma 2.1 is proved. □

We will consider, in the sequel, several numbers such as N, q and r that will go to infinity. It will be very important the order we take them, that is, which number goes to infinity first, and then which one will be the next, etc... However, the value K_0 will be much larger than all of them. Lemma 2.1 says that $h > K_0$ has small order in the computation of the spectral density function.

Let x_0 be a point in a set of μ-probability one. From now on we will denote $T^t(x_0) \equiv x_t$ and $X_t = \varphi(T^t(x_0)) \equiv \varphi(x_t)$.

From expression (1.3) one has, for N fixed, that

$$I(\lambda_k) = \frac{1}{4\pi^2 N}\sum_{s,t=1}^{N} X_t X_s e^{-i\lambda_k(t-s)} = \frac{1}{4\pi^2 N}\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N}} X_s X_{s+h} e^{-i\lambda_k h}, \tag{2.3}$$

where in the last equality we change variable $t-s$ by h. Therefore, for each s fixed, the range of h is $1-s \leq h \leq N-s$.

We want to prove the convergence in distribution of the periodogram (to the function $f_X(\cdot)$) based on a time series $\{X_t\}_{t=1}^N$ of the process $\{X_t\}_{t\in\mathbb{N}}$ given by (1.1) beginning with a μ-almost everywhere point x_0.

The proof of Theorem 1.1 will be given after some lemmas. In order to have the convergence in distribution sense we will show that for any smooth function $g : S \to \mathbb{R}$

$$\begin{aligned} \lim_{N\to\infty}\left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g\right\rangle &= \langle f_X, g\rangle = \int_0^{2\pi} f_X(\lambda)g(\lambda)d\lambda = \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{h\in\mathbb{N}}\gamma_X(h)e^{-i\lambda h}g(\lambda)d\lambda, \end{aligned}$$

where

$$\left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle \equiv \int_0^{2\pi} g(x)\frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k} =$$

$$= \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)g(\lambda_k) =$$

$$= \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k)\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N}} X_s X_{s+h} e^{-i\lambda_k h}.$$

Remark 2.1. Given x, in order to determine the (approximated) value $f_X(x)$, one takes a continuous function g such that it has support in a small neighborhood of x (see pages 78-79 in [8]) and apply the theorem for such g. Changing the point x one can graph the (approximated) function $f_X(x)$.

Let $r \in \mathbb{N} - \{0\}$ be a fixed value such that Lemma 2.1 holds. Then, we can write

$$\left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle = \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k)\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N}} X_s X_{s+h} e^{-i\lambda_k h} \tag{2.4}$$

$$= \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k)\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N\\ |h|\leq r}} X_s X_{s+h} e^{-i\lambda_k h} \tag{2.5}$$

$$+ \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k)\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N\\ |h|> r}} X_s X_{s+h} e^{-i\lambda_k h}. \tag{2.6}$$

Lemma 2.2 below is crucial and we shall prove that the expression (2.6) goes to zero, when $N \to \infty$.

Lemma 2.2. *Given $\epsilon_1 > 0$, there exists r such that, for all $K_0 > r$ and for $x_0 \in [0,1)$ μ-almost everywhere, there exists $N_1 \in \mathbb{N} - \{0\}$ such that*

$$\left| \frac{1}{4\pi^2N^2}\sum_{k=0}^{N} g(\lambda_k)\sum_{\substack{1\leq s\leq N\\ 1\leq s+h\leq N\\ K_0>|h|> r}} X_s X_{s+h} e^{-i\lambda_k h} \right| < \epsilon_1, \ \textit{for all } N > N_1.$$

Proof: Given $\epsilon_1 > 0$, let ϵ be such that

$$\epsilon = \frac{4\pi^2}{2M_1}\epsilon_1.$$

Given r and K_0 fixed, the function

$$v(x) \equiv \sum_{r<|h|<K_0} \left|\varphi(x)\varphi(T^h(x))\right| \left|e^{-i\lambda_k h}\right| = \sum_{r<|h|<K_0} \left|\varphi(x)\varphi(T^h(x))\right|$$

is continuous. If r is large enough one has, for this $\epsilon > 0$ and $K_0 > r$, that

$$\left|\int v(x)d\mu(x)\right| < \frac{\epsilon}{3}. \tag{2.7}$$

Here we use again that $\sum_{h\in\mathbb{N}} |\gamma_X(h)|$ converges, since the autocovariance function of order h of the process $\{X_t\}_{t\in\mathbb{N}}$ goes to zero with order of convergence $|h|^{-\beta}$, with $\beta > 2$.

Now we fix r and K_0 (much more larger than r). For such fixed function $v(\cdot)$, we want to estimate the μ-measure of "bad" points x_0 given by

$$P_{N_0}^{\frac{\epsilon}{3}} = \mu\left(\left\{x_0;\ \sup_{N>N_0}\left|\frac{1}{N}\sum_{s=1}^{N} v(T^s(x_0)) - \int v(x)d\mu(x)\right| > \frac{\epsilon}{3}\right\}\right).$$

From Theorem 13, part 1, in [5], as $\sigma_f(-\delta,\delta) = o(\delta^{\beta-1})$ as $\delta \to 0$ (since *Assumption* given by (1.4) holds), then $P_{N_0}^{\frac{\epsilon}{3}} = o(N_0^{-(\beta-1)})$ as $N_0 \to \infty$. Therefore, as $\beta > 2$,

$$\sum_{N_0=1}^{\infty} P_{N_0}^{\frac{\epsilon}{3}} < \infty.$$

Then, from Borel-Cantelli Lemma, one has that

$$\mu\left(\left\{x_0;\ \sup_{N>N_0}\left|\frac{1}{N}\sum_{s=1}^{N} v(T^s(x_0)) - \int v(x)d\mu(x)\right| > \frac{\epsilon}{3}\right\}\ \ i.o.\right) = 0,$$

that, is, for any x_0 μ-almost everywhere, there exists $N_1 = N_1(x_0) > 0$ such that $x_0 \notin P_{N_0}^{\frac{\epsilon}{3}}$, for all $N_0 > N_1$. Hence,

$$\sup_{N>N_0}\left|\frac{1}{N}\sum_{s=1}^{N} v(T^s(x_0)) - \int v(x)d\mu(x)\right| \leq \frac{\epsilon}{3}. \tag{2.8}$$

From the expressions (2.7) and (2.8), for all $N_0 > N_1$ and for all $N > N_0$, we have

$$\left|\frac{1}{N}\sum_{s=1}^{N} v(T^s(x_0))\right| = \left|\frac{1}{N}\sum_{1\leq s\leq N}\ \sum_{r<|h|<K_0} \varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right| < \frac{2\epsilon}{3}.$$

This is not enough. As r and K_0 are fixed, one has

$$\lim_{N\to\infty}\frac{1}{N}\sum_{s=0}^{1-h}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))=0$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{s=N-h}^{N}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))=0.$$

Therefore, given $\frac{\epsilon}{3}>0$, there exists $N_2\in\mathbb{N}-\{0\}$ such that, for all $N>N_2$,

$$\left|\frac{1}{N}\sum_{s=0}^{1-h}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|<\frac{\epsilon}{6}$$

and

$$\left|\frac{1}{N}\sum_{s=N-h}^{N}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|<\frac{\epsilon}{6}.$$

Since

$$\left|\frac{1}{N}\sum_{\substack{1\le s\le N\\ 1\le s+h\le N}}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|=$$

$$=\left|\frac{1}{N}\sum_{1\le s\le N}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))-\frac{1}{N}\sum_{\substack{1\le s\le N\\ s<1-h}}\sum_{r<|h|<K_0}\right.$$

$$\left.\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))-\frac{1}{N}\sum_{\substack{1\le s\le N\\ s>N-h}}\sum_{r<|h|<K_0}\varphi(T^s(x_0))\varphi(T^{s+h}(x_0))\right|<$$

$$<\frac{2\epsilon}{3}+\frac{\epsilon}{6}+\frac{\epsilon}{6}=\epsilon.$$

Since $M_1=\sup_{\lambda\in[0,2\pi)}|g(\lambda)|$, one has, for large N, that

$$\left|\frac{1}{4\pi^2N}\sum_{k=0}^{N}g(\lambda_k)\frac{1}{N}\sum_{\substack{1\le s\le N\\ 1\le s+h\le N\\ r<|h|<K_0}}X_sX_{s+h}e^{-i\lambda_k h}\right|\le\frac{1}{4\pi^2N}\sum_{k=0}^{N}|g(\lambda_k)|\,\epsilon<$$

$$<\frac{N+1}{4\pi^2N}M_1\epsilon<\frac{2M_1}{4\pi^2}\epsilon=\epsilon_1.$$

Therefore, Lemma 2.2 is proved. $\square$

Remark 2.2. Consider ϵ fixed and r_0 as in Lemma 2.1. We can truncate the infinite sum defining the spectral density function in K_0 according to Lemma 2.1. Now in Lemma 2.2, for $\epsilon_1 = \epsilon/3$ take $r > r_0$ and $K_0 > r$. In this way we obtain in Lemma 2.2 a value N_1 and all values $N > 0$, in the future, will be larger than such N_1. The values r and K_0 will be fixed in the future but we still have to impose some more restrictions. The above lemma will imply, as we will see later, that the values h such that $r < |h| < K_0$ have small order in the estimation of the values of the periodogram function.

Now we return to the proof of the main theorem. We have to analyze the contribution of the values $|h| < r$.

From Lemma 2.2, the equality in expression (2.4), when $N \to \infty$, can be rewritten as

$$\left\langle \frac{1}{N} \sum_{h=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle = \frac{1}{4\pi^2 N^2} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N \\ 1\le s+h\le N \\ |h|\le r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1). \tag{2.9}$$

Now, we fix q and let us define $B_1, \cdots, B_q$ a partition of the unit interval. The intervals B_j, $j \in \{1, \cdots, q\}$, are such that $B_i \cap B_j = \phi$, for all $i, j \in \{1, \cdots, q\}$, $i \neq j$, and such that $\bigcup_{j=1}^{q} B_j = [0, 1)$, where $B_j = \left[\frac{j}{q}, \frac{j+1}{q}\right)$, for $j \in \{1, \cdots, q\}$.

Remark 2.3. The value $X_{s+h} = \varphi(T^{s+h})(x_0)$ is defined above for positive $s+h$. However, the value h can be negative. In order to avoid a heavy notation, when h is negative, the expression $\varphi(x_s)\varphi(T^h(x_s))$ in the sum below will mean $\varphi(x_{s+h})\varphi(T^{-h}(x_{s+h}))$.

Let α_j be a fixed interior point of B_j, for $j \in \{1, \cdots, q\}$. Then, the expression (2.9) can be rewritten as

$$\begin{aligned}
\left\langle \frac{1}{N} \sum_{h=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle &= \frac{1}{4\pi^2 N^2} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N \\ 1\le s+h\le N \\ |h|\le r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1) = \\
= \quad & \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \frac{1}{N} \sum_{\substack{1\le s\le N \\ 1\le s+h\le N \\ |h|\le r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1) = \\
= \quad & \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{j=1}^{q} \frac{1}{N} \sum_{\substack{s: x_s \in B_j \\ 1\le s\le N \\ 1\le s+h\le N \\ |h|\le r}} \left[\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))\right] e^{-i\lambda_k h} \\
+ \quad & \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\le s\le N \\ 1\le s+h\le N \\ |h|\le r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \qquad (2.10) \\
+ \quad & o(1),
\end{aligned}$$

where $X_s = \varphi(X_{s-1}) = \varphi(T^s(X_0))$ and $x_s = T^s(x_0)$.

Note that the restriction $1 \leq s+h \leq N$, with $|h| < r$ and r fixed, is a mild assumption because the number of s such that $1-h < s < N-h$ is of the same order as N. By Birkoff's Theorem

$$\lim_{N\to\infty} \frac{1}{N} \# \left[\begin{array}{c} x_s \in B_j \\ 1 \leq s+h \leq N \end{array} \right] = \lim_{N\to\infty} \frac{1}{N} \#[x_s \in B_j] = \mu(B_j),$$

given $\epsilon > 0$ take N large enough such that

$$\left| \frac{1}{N} \#[x_s \in B_j] - \mu(B_j) \right| \leq \frac{4\pi^2}{2M_1 r q M_2^2} \epsilon,$$

for any $j \in \{1, \cdots, q\}$, where $M_1 = \sup_{\lambda\in[0,2\pi]} |g(\lambda)|$ and $M_2 = \sup_{x\in S} |\varphi(x)|$. Suppose N goes to infinity faster than q.

Now, we will show the following claim:

Claim 2.1. *Given $\epsilon > 0$, for N and r large enough, the absolute value of the expression (2.10) can be written as*

$$\left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\leq s\leq N \\ 1\leq s+h\leq N \\ |h|\leq r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| <$$

$$< \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j) \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| +$$

$$+ \frac{N+1}{N} \epsilon, \tag{2.11}$$

for large N.

Proof. Observe that using Birkoff's Theorem, for large N,

$$\left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\leq s\leq N \\ 1\leq s+h\leq N \\ |h|\leq r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| \leq$$

$$\leq \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j) \varphi(\alpha_j) \varphi(T^h(\alpha_j)) e^{-i\lambda_k h} \right| +$$

$$+ \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} |g(\lambda_k)| \sum_{|h|\leq r} \sum_{j=1}^{q} \frac{4\pi^2 \epsilon}{2M_1 r q M_2^2} |\varphi(\alpha_j)| \left|\varphi(T^h(\alpha_j))\right| \leq$$

$$\leq \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} \right| +$$

$$+\frac{1}{N}\sum_{k=0}^{N} |g(\lambda_k)| \sum_{|h|\leq r} \sum_{j=1}^{q} \frac{\epsilon}{2M_1 rq M_2^2} M_2^2 =$$

$$= \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} \right| +$$

$$+\frac{1}{N}\sum_{k=0}^{N} |g(\lambda_k)| 2rq \frac{\epsilon}{2M_1 rq} <$$

$$< \left| \frac{1}{4\pi^2} \frac{1}{N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} \right| + \frac{N+1}{N}\epsilon.$$

This proves Claim 2.1. $\square$

As

$$\left\langle \frac{1}{N}\sum_{h=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle = \frac{1}{4\pi^2 N^2}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\leq s\leq N \\ 1\leq s+h\leq N \\ |h|\leq r}} X_s X_{s+h} e^{-i\lambda_k h} + o(1) =$$

$$= \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{j=1}^{q} \frac{1}{N} \sum_{\substack{s: x_s \in B_j \\ 1\leq s\leq N \\ 1\leq s+h\leq N \\ |h|\leq r}} \left[\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))\right] e^{-i\lambda_k h}$$

$$+ \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{\substack{1\leq s\leq N \\ 1\leq s+h\leq N \\ |h|\leq r}} \sum_{j=1}^{q} \frac{1}{N} \#[x_s \in B_j] \varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} + o(1),$$

from Claim (2.1), we can write

$$\left\langle \frac{1}{N}\sum_{h=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle =$$

$$= \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{j=1}^{q} \sum_{\substack{x_s \in B_j \\ 1\leq s\leq N \\ 1\leq s+h\leq N \\ |h|\leq r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))] e^{-i\lambda_k h}$$

$$+ \frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} + o(1). \qquad (2.12)$$

In Lemma 2.3 we shall prove that the first term in the expression (2.12) can be taken as small as we want if N and q are large enough.

Lemma 2.3. *Given $\epsilon > 0$,*

$$\frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k)\sum_{j=1}^{q}\frac{1}{N}\Big(\sum_{\substack{x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ 0<h\le r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h} +$$

$$\sum_{\substack{x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ -r\le h\le 0}} [\varphi(x_{s+h})\varphi(T^{-h}(x_{s+h})) - \varphi(\alpha_j)\varphi(T^{-h}(\alpha_j))]e^{-i\lambda_k h}\Big) < \epsilon, \qquad (2.13)$$

for q sufficiently large but fixed and for all N large enough.

Proof. In order to avoid a heavy notation we consider just the part with $h > 0$ in expression (2.13). The reasoning for the case $h < 0$ is similar. Since φ is a continuous map on the compact set $[0, 1)$ it is uniformly continuous. The transformation T is also continuous, so $\varphi(\cdot)\varphi(T^h(\cdot))$ is also a uniformly continuous function. Therefore, for fixed $h \in \mathbb{N}$, for all $\epsilon_1 > 0$, there exists δ_h such that for all x, y with $|x-y| < \delta_h$, then

$$|\varphi(x)\varphi(T^h(x)) - \varphi(y)\varphi(T^h(y))| < \epsilon_1.$$

Since $0 \le h \le r$ is uniformly bounded one can find a uniform δ that works for all $h \in \mathbb{N}$ such that $0 \le h \le r$.

Take $\epsilon_1 = \frac{4\pi^2}{2M_1}\epsilon > 0$ and q large enough such that $\frac{1}{q} < \delta$. Therefore, if length of $B_j < \delta$, for all $j \in \{1, \cdots, q\}$, we have

$$|\varphi(x)\varphi(T^h(x)) - \varphi(y)\varphi(T^h(y))| < \epsilon_1,$$

for all $x, y \in B_j$ and for any $h \in \mathbb{Z}$ such $|h| < r$.

Therefore,

$$\left|\frac{1}{4\pi^2}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k)\sum_{j=1}^{q}\frac{1}{N}\sum_{\substack{x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} [\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h}\right| \le$$

$$\le \frac{1}{4\pi^2 N}\sum_{k=0}^{N} |g(\lambda_k)|\frac{1}{N}\sum_{j=1}^{q}\sum_{\substack{x_s\in B_j\\ 1\le s\le N\\ 1\le s+h\le N\\ |h|\le r}} \big|\varphi(x_s)\varphi(T^h(x_s)) - \varphi(\alpha_j)\varphi(T^h(\alpha_j))\big| \le$$

$$\leq \frac{1}{4\pi^2 N}\sum_{k=0}^{N}|g(\lambda_k)|\frac{1}{N}\sum_{j=1}^{q}\sum_{\substack{x_s\in B_j\\1\leq s\leq N\\1\leq s+h\leq N\\|h|<r}}\epsilon_1 = \frac{1}{4\pi^2 N}\sum_{k=0}^{N}|g(\lambda_k)|\frac{1}{N}\sum_{\substack{1\leq s\leq N\\1\leq s+h\leq N\\|h|<r}}\epsilon_1,$$

for x_s and α_j such that $|x_s-\alpha_j|<\delta$, where the double summation, in the above expression, has N terms all of them less than ϵ_1. Therefore,

$$\frac{1}{4\pi^2 N}\sum_{k=0}^{N}g(\lambda_k)\sum_{j=1}^{q}\frac{1}{N}\sum_{\substack{x_s\in B_j\\1\leq s\leq N\\1\leq s+h\leq N\\|h|\leq r}}[\varphi(x_s)\varphi(T^h(x_s))-\varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h} <$$

$$< \frac{(N+1)M_1}{4\pi^2 N}\epsilon_1 < \frac{2M_1}{4\pi^2}\epsilon_1 = \epsilon,$$

where

$$M_1 = \sup_{\lambda\in(0,2\pi]}|g(\lambda)|.$$

Therefore, Lemma 2.3 is proved. □

Now, using Lemma 2.3, the whole expression (2.12) can be rewritten as

$$\begin{aligned}\left\langle \frac{1}{N}\sum_{k=0}^{N}I(\lambda_k)\delta_{\lambda_k}, g\right\rangle &= \\ = \quad & \frac{1}{4\pi^2 N}\sum_{k=0}^{N}g(\lambda_k)\sum_{|h|\leq r}\sum_{j=1}^{q}\mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} \qquad (2.14)\\ + \quad & o(1).\end{aligned}$$

The following lemma proves that the expression (2.14) goes to $\langle f_{X,r}, g\rangle$, when $N\to\infty$.

Lemma 2.4. *For r fixed,*

$$\frac{1}{4\pi^2 N}\sum_{k=0}^{N}g(\lambda_k)\sum_{|h|\leq r}\sum_{j=1}^{q}\mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))]e^{-i\lambda_k h}\to\langle f_{X,r}, g\rangle\,,$$

when $q\to\infty$ and $N-q\to\infty$.

Proof. Given $\epsilon_1>0$, note that for fixed N, with N much larger than q, where q is also large, one has

$$\left|\sum_{|h|\leq r}\sum_{j=1}^{q}\mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j))e^{-i\lambda_k h} - \sum_{|h|\leq r}\left(\int_0^1\varphi(x)\varphi(T^h(x))d\mu(x)\right)e^{-i\lambda_k h}\right|$$

$$\leq \sum_{|h|\leq r} \frac{\epsilon_1}{2r} = \epsilon_1.$$

This is true since, for fixed h and $|h| < r$,

$$\sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) \to \int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x), \quad \text{when } q \to \infty.$$

Therefore, given $\epsilon_1 > 0$, if q is large then, for all $|h| < r$,

$$\left| \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) - \int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x) \right| < \epsilon_1.$$

Since

$$\int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x) = \gamma_X(h), \quad \text{for fixed } h,$$

and

$$f_{X,r}(\lambda_k) = \frac{1}{2\pi} \sum_{|h|\leq r} \gamma_X(h) e^{-i\lambda_k h},$$

one has

$$\left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} - \right.$$

$$\left. - \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \left(\int_0^1 \varphi(x)\varphi(T^h(x))d\mu(x) \right) e^{-i\lambda_k h} \right| \leq$$

$$\leq \left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) e^{-i\lambda_k h} \right| \epsilon_1,$$

for large q. Therefore, one has

$$\left| \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k) \sum_{|h|\leq r} \sum_{j=1}^{q} \mu(B_j)\varphi(\alpha_j)\varphi(T^h(\alpha_j)) e^{-i\lambda_k h} - \right.$$

$$\left. - \frac{1}{2\pi N} \sum_{k=0}^{N} g(\lambda_k) f_{X,r}(\lambda_k) \right| \leq \frac{1}{4\pi^2 N} \sum_{k=0}^{N} |g(\lambda_k)| \epsilon_1, \tag{2.15}$$

for large q, and N much larger than q. Expression (2.15) suggests to take

$$\epsilon_1 = \frac{2\pi^2}{M_1} \epsilon > 0.$$

Then,

$$\left\langle \frac{1}{N} \sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle = \frac{1}{2\pi N} \sum_{k=0}^{N} g(\lambda_k) f_{X,r}(\lambda_k) + \frac{1}{4\pi^2 N} \sum_{k=0}^{N} g(\lambda_k)\epsilon_1 + o(1)$$

$$= \frac{1}{2\pi}\frac{1}{N}\sum_{k=0}^{N} g(\lambda_k) f_{X,r}(\lambda_k) + \frac{N+1}{N}\epsilon + o(1)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} g(\lambda) f_{X,r}(\lambda) d\lambda + \frac{N+1}{N}\epsilon + o(1),$$

for large N.
Then, for large N, one has

$$\left| \left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle - \frac{1}{2\pi}\int_0^{2\pi} f_{X,r}(\lambda) g(\lambda) d\lambda \right| < o(1). \tag{2.16}$$

This proves Lemma 2.4. □

Now we shall prove Theorem 1.1. Considering the expression (2.16) and Lemma 2.1, for given $\epsilon/3 > 0$, one has

$$\left| \left\langle \frac{1}{N}\sum_{k=0}^{N} I(\lambda_k)\delta_{\lambda_k}, g \right\rangle - \frac{1}{2\pi}\int_0^{2\pi} f_X(\lambda) g(\lambda) d\lambda \right| \leq \epsilon,$$

for N large enough.

This proves our main theorem. □

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