

# Poncelet pairs and the Twist Map associated to the Poncelet Billiard

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## Abstract

Consider a fixed differentiable curve  $K$  (which is the boundary of a convex domain) and a family indexed by  $\lambda \in [0, 1]$  of variables differentiable curves  $L$  such that are the boundary of convex domains  $A_\lambda$ . Suppose that for  $\lambda_1 < \lambda_2$  we have  $A_{\lambda_1} \subset A_{\lambda_2}$ . Then, the number of  $n$ -Poncelet pairs is given by  $\frac{e(n)}{2}$ , where  $e(n)$  is the number of natural numbers  $m$  smaller than  $n$  and which satisfies  $\text{mcd}(m, n) = 1$ . The curve  $K$  do not have to be part of the family.

In order to show this result we consider an associated billiard transformation and a twist map which preserves area.

We use Aubry-Mather theory and the rotation number of invariant curves to obtain our main result.

In the last section we estimate the derivative of the rotation number of a general twist map using some properties of the continued fraction expansion .

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# 1 Introduction - The Poncelet Billiard

All results here are for a fixed differentiable curve  $K$  (which is the boundary of a convex domain) and a family indexed by  $\lambda \in [0, 1]$  of variables differentiable  $C^\infty$  curves  $L$  such that are the boundary of convex domains  $A_\lambda$ . Suppose that for  $\lambda_1 < \lambda_2$  we have  $A_{\lambda_1} \subset A_{\lambda_2}$ .

Some explicit results (not for the Poncelet counting problem) for billiards in the case the curves are ellipses were obtained in [c1]. Our reasoning apply for a more general family of curves. We refer the reader to [c2] for some applications of the Poncelet billiard problem.

In order to simplify the exposition we will refer to circles instead of curves. Some computations are explicit for circles but our reasoning applies to the setting we just described above. The estimates of Proposition 2 bellow are not contained in previous results which analyze the Poncelet problem.

Let  $K$  a circle as in figure 1, and also  $L$  another circle interior to  $K$  with  $A, A'$  variable points in  $K$ .

Consider the Poncelet transformation associated to such pair of circles  $K$  and  $L$  and the corresponding image of the point  $A$  being  $A'$ , and then the image of  $A'$  is  $A''$  and so on... (see figure 1). A nice description of the problem appears in [K]and [T].

$T, C, o$  are shown in figure 2. Here  $R$  is the radius the radius of the circle  $K$  and  $C$  is the center of  $K$ .

We will use the variables  $\theta$  and  $\varphi$  to describe the point  $A$  and his future hit as in figure 1 and 2. The point  $A'$  has coordinates  $\theta'$  and  $\varphi'$ .

The variable points  $B, B'$  on the  $x$ -axis are also described in figure 2.

Denote  $\theta' = G_1(\theta, \varphi)$  and  $\varphi' = G_2(\theta, \varphi)$ . The transformation  $G = (G_1, G_2)$  can be consider also as a transformation of  $(A, r) \rightarrow (A', r')$ , where  $r$  and  $r'$  are tangent lines to the circles as in figure 1.

Note that once  $A$  is fixed, if we take a variable line  $r$ , this means we are considering different circles  $L$ . The circle  $K$  is considered fixed in our setting.

The reader can find several references about the Poncelet Billiard in [1], [3], [4], [6], [7], [8], [10].

We say  $K, L$  is a  $n$ -Poncelet pair if there exists an  $A$  such that the successive iteration of the procedure described above, after  $n$  steps, returns to  $A$ .

The existence of a  $n$ -periodic point  $(A, r)$  for  $G$  (for a certain pair of circles  $K, L$ ) is equivalent to the existence of a  $n$ -Poncelet pair. We would like to count the number of possible Poncelet pairs of order  $n$ .

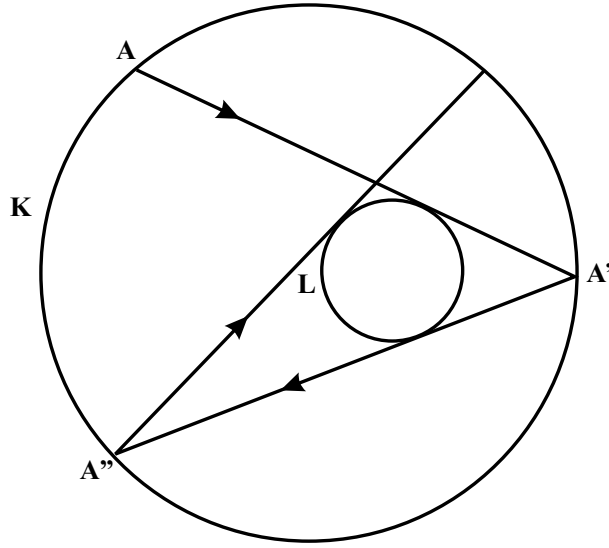


Figure 1:

It is easy to obtain the analytic expression

$$G(\theta, \varphi) = (2\varphi - \theta + \pi, 3\varphi - 2\theta - B(\theta') + \pi),$$

where  $B(\theta') = 2 \arctan\left(\frac{C \sin(\theta')}{R+C \cos(\theta')}\right)$ .

We know that  $r$  is a periodic function of  $\varphi$  of period  $\pi$ . We change coordinates to get  $x = \frac{\theta}{2\pi}$  and  $y = \frac{\varphi}{\pi}$ .

Therefore, denoting by  $f$  the transformation  $G$  in the new coordinates

$$(x', y') = f(x, y) = (y - x + 1/2, 3y - 4x + Z(y - x + 1/2) + 1),$$

where  $Z(s) = -B(2\pi s)$ . The function  $Z$  is periodic of period 1.

Therefore  $f = (f_1, f_2)$  is a twist map on the torus (see [2],[5],[9] for references), because

$$1) \quad \frac{\partial f_1(x, y)}{\partial y} = \frac{\partial x'}{\partial y} = 1 > 0,$$

and, also preserves area

$$2) \quad \text{Jac } Df = \text{Det} \begin{pmatrix} -1 & 1 \\ -4 - Z'(y - x + 1/2) & 3 + Z'(y - x + 1/2) \end{pmatrix} = 1.$$

One can write  $y = x' + x - 1/2$ , and therefore the generating potential  $h(x, x')$  for such  $f$  is

$$h(x, x') = -x x' - \frac{x^2 - x}{2} + \frac{3(x')^2 - x'}{2} + H(x'),$$

where  $H$  is such that  $H'(x') = Z(x')$ .

The twist map is not exact.

After this brief introduction we will present in next section a sequence of results that will be used in the last section to show our main theorem:

**Theorem 1.** *For a fixed circle  $K$  and for a family of variables concentric circles  $L$ , the number of  $n$ -Poncelet pairs is  $\frac{e(n)}{2}$ , where  $e(n)$  is the number of natural numbers  $m$  smaller than  $n$  and which satisfies  $\text{mcd}(m, n) = 1$ .*

The general problem for conics is the following: given a conic  $K$  and a pencil of conics  $\mathcal{L}$ , obtain the number of conics  $L \in \mathcal{L}$  such that  $K, L$  is a  $n$ -Poncelet pair. This problem was solved in [11] for the complex plane for a generic pencil but under the condition that  $K \in \mathcal{L}$ . In our analysis we have that  $K$  is not in  $\mathcal{L}$ .

Consider a family  $g_t : \mathbb{R} \rightarrow \mathbb{R}$ , where  $a \leq t \leq b$ , of monotonous increasing homeomorphisms, such that

$$g_t(x + 1) = g_t(x) + 1, \quad x \in \mathbb{R}, \quad a \leq t \leq b.$$

From the identification  $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$ , from the family  $g_t$ , we obtain another one  $f_t : S^1 \rightarrow S^1$ , where  $a \leq t \leq b$ , of homeomorphisms of the circle which preserve orientation.

We know that is well defined

$$r(t) = \lim_{k \rightarrow \infty} \frac{g_t^k(x) - x}{k}, \quad a \leq t \leq b,$$

and the value  $r(t)$  indeed is independent of  $x$ .

We are interested in properties of the function  $r(t)$  and of the rotation number function  $\rho(t) = r(t) \pmod{1} \in \frac{\mathbb{R}}{\mathbb{Z}}$ .

In the last section we show

**Proposition 1:** The function  $r : [a, b] \rightarrow \mathbb{R}$  is continuous monotonous increasing. Moreover, if  $a \leq t_1 < t_2 \leq b$  and  $r(t_1) \notin \mathbb{Q}$  or  $r(t_2) \notin \mathbb{Q}$ , then  $r(t_1) < r(t_2)$ .

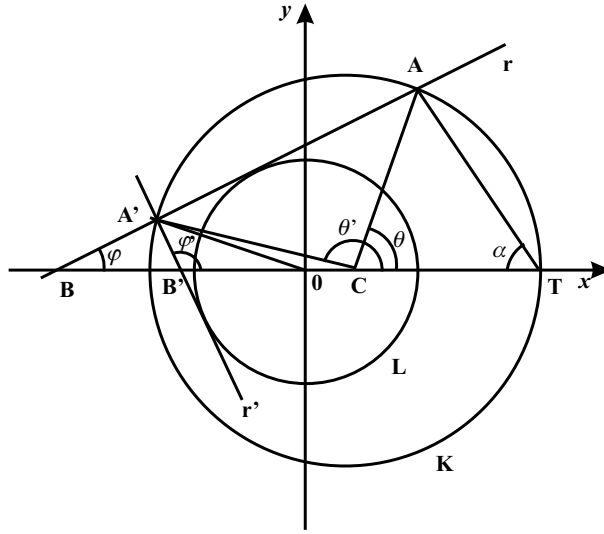


Figure 2:

And also,

**Proposition 2:** There exist a set of full measure  $X \subset \mathbb{R}$  (that is,  $\mathbb{R} - X$  has Lebesgue measure zero) and such that if  $\tau \in (a, b)$  and  $r(\tau) \in X$ , then

$$\limsup_{t_2 \rightarrow \tau^-, t_1 \rightarrow \tau^+} \frac{r(t_2) - r(t_1)}{(t_2 - t_1)^2} \geq \frac{m^2}{e^{2F} (1 + e^{2F})^2}.$$

## 2 Results for Twist Maps

Consider  $f : S^1 \rightarrow S^1$ , where  $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$ , a circle homeomorphism which preserves orientation. Denote by  $g : \mathbb{R} \rightarrow \mathbb{R}$  a lifting of  $f$ .

Denote by  $r(g)$  the limit

$$\lim_{k \rightarrow \infty} \frac{g^k(x) - x}{k} = r(g),$$

which is independent of  $x \in \mathbb{R}$  (see [9] chapter 11).

The rotation number of  $f$ , denoted by  $\rho(f)$ , is the number  $r(g) \pmod{1} \in S^1$ .

General references for the rotation number and Twist Maps are [2], [5], [9].

**Lemma 2.** For  $f$  and its lifting  $g$ :

a)  $\rho(f)$  is an invariant of conjugation (by an homeomorphism which preserves orientation) for  $f$ ,

b) if  $\rho(f) = \frac{p}{q} \pmod{1}$ , where  $q > 0$  and  $p, q$  are relatively prime, then there exists a  $x_0$  such that  $g^q(x_0) = x_0 + p$  (see [9]), and the orbit of  $x_0 \pmod{1}$  is periodic of period exactly  $q$ .

The following Lemma is also well known:

**Lemma 3.** Consider  $g, h : \mathbb{R} \rightarrow \mathbb{R}$  continuous and strictly increasing such that  $g(x + 1) = g(x) + 1$  and  $h(x + 1) = h(x) + 1$ ,  $\forall x \in \mathbb{R}$ .

Then,

a) If  $h(x) \leq g(x)$ ,  $\forall x \in \mathbb{R}$ , then  $r(h) \leq r(g)$ .

b)  $g \rightarrow r(g)$  is continuous in the  $C^0$  topology.

c) If  $h(x) < g(x)$ ,  $\forall x \in \mathbb{R}$ , and if  $r(h)$  or  $r(g)$  is irrational, then  $r(h) < r(g)$ .

Now, consider the cylinder  $X = S^1 \times \mathbb{R} = \frac{\mathbb{R}}{\mathbb{Z}} \times \mathbb{R}$  and the natural projection  $\pi : \mathbb{R}^2 \rightarrow X$ .

**Definition 4.** The vector field  $u$  over the cylinder  $X$  is called the ascendent vector field, if  $u$  is such that  $u(a, b) =$  the tangent vector to the curve  $t \rightarrow (a, t + b)$ , in the point  $t = 0$ .

We denote by  $\pi_1 : X \rightarrow S^1$  and  $\pi_2 : X \rightarrow \mathbb{R}$  the natural projections.

We will consider here diffeomorphisms of class  $C^1$ ,  $f : X \rightarrow X$ , such that

a)  $f$  preserves orientation and the ends of  $X$ ,

b)  $f$  preserves area of a Riemannian metric, and

c) for each point  $x \in X$  the vectors  $u$  and  $df_x(u)$  are independent and coherent with the natural orientation.

Consider  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $F(x, y) = (F_1(x, y), F_2(x, y))$ , a lifting of such  $f$ .

Then  $F$  satisfies the conditions

1)  $F_1(x + 1, y) = F_1(x, y) + 1$ ,

2)  $F_2(x + 1, y) = F_2(x, y)$ ,

3) for each  $x$ , we have  $\lim_{y \rightarrow \infty} F_2(x, y) = \infty$ ,

4)  $F$  preserves area of a Riemannian metric which is invariant by translation  $(x, y) \rightarrow (x + 1, y)$ ,

5)  $\frac{\partial F_1}{\partial y}(x, y) > 0$  (the twist condition).

We point out that by condition 5) we have that given  $(x_0, y_0)$  and  $(x_0, y_1)$  in  $\mathbb{R}^2$ , such that,  $y_1 > y_0$ , then  $F_1(x_0, y_1) > F_1(x_0, y_0)$ .

**Definition 5.** Consider  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , a continuous function of period 1, and the curve  $\Gamma \subset X$  with parametrization  $x \rightarrow \pi(x, \phi(x))$ ,  $0 \leq x \leq 1$ . We say  $\Gamma$  is an invariant rotational circle for  $f$  if  $f(\Gamma) = \Gamma$ .

We point out that  $\Gamma$  is an oriented circle and  $\pi^{-1}(\Gamma)$  is the graph of  $\phi$ . Moreover  $f : \Gamma \rightarrow \Gamma$  preserves the orientation of  $\Gamma$ . We denote by  $\rho(\Gamma)$  the rotation number of  $\rho(f|\Gamma)$ . The transformation  $F$  also preserves the graph of  $\phi$  and its orientation.

The usual definition of invariant rotational circle is more general ([2] chapter 3, definition 11) but a theorem due to Birkhoff ([2], 3.1) shows that, under quite general conditions, this definition coincides with our definition 5. The rotational invariant curve we are going to consider here (a subset of the torus) is a set of positions  $q$  on the circle  $K$  and respective tangent line tangent to the small circle  $L$  (passing through  $q, p$ ) as shown in figure 3.

Consider a  $g$  associated to  $\Gamma$  by  $g(x) = F_1(x, \phi(x)), \forall x \in \mathbb{R}$ . The function  $g$  is continuous strictly increasing and satisfies

$$g(x + 1) = F_1(x + 1, \phi(x + 1)) = F_1(x + 1, \phi(x)) = g(x) + 1$$

We claim that

$$\rho(\Gamma) = r(g) \pmod{1}.$$

Indeed, by definition  $\pi_1|_{\Gamma} : \Gamma \rightarrow S^1$  is a homeomorphism which preserves orientation. Consider  $\varphi = (\pi_1|_{\Gamma}) \circ (f|\Gamma) \circ (\pi_1|_{\Gamma})^{-1}$ . Then  $\varphi$  preserves orientation and  $\rho(\Gamma) = \rho(\varphi)$ .

By the other side

$$\pi_1 \circ \pi(x, 0) = x \pmod{1}, \forall x \in \mathbb{R}.$$

As

$$\begin{aligned} \varphi(\pi_1(\pi(x, 0))) &= \varphi(\pi_1(\pi(x, \phi(x)))) = \\ \pi_1(f(\pi(x, \phi(x)))) &= \pi_1(\pi(F(x, \phi(x)))) = \pi_1 \circ \pi(g(x), 0), \end{aligned}$$

then  $g$  is a lifting of  $\varphi$ . Then,  $\rho(\varphi) = r(g) \pmod{1}$  and the claim is true.

**Theorem 6.** Consider  $\Gamma_1$  and  $\Gamma_2$  two rotational invariant circles associated respectively to  $\phi_1$  and  $\phi_2$ .

Suppose  $\phi_1(x) < \phi_2(x), \forall x \in \mathbb{R}$ . Denote

$$g_1(x) = F_1(x, \phi_1(x)), \quad g_2(x) = F_1(x, \phi_2(x)), \quad \forall x \in \mathbb{R}.$$

Then

$$r(g_1) < r(g_2).$$

**Proof:** Following [2] 3.3, from  $\phi_1 < \phi_2$  we have that that  $g_1(x) < g_2(x), \forall x \in \mathbb{R}$ . Therefore,  $r(g_1) \leq r(g_2)$ . If  $r(g_1)$  or  $r(g_2)$  is irrational, then the claim is true (Lemma 3).

Suppose, that  $r(g_1) = r(g_2) = \frac{p}{q}$ , where  $q > 0$  and  $p, q$  are relatively prime.

Consider the translation  $T_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $T_n(x, y) = (x + n, y)$  and the diffeomorphism  $G = T_{-p} \circ F^q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . We will show that  $G(A)$ , for a certain region  $A$ , is strictly inside the set  $A$  and we will get a contradiction with the fact that  $G$  preserves area.

We denote  $m' = G(m), m \in \mathbb{R}^2$ .

Note that  $G$  preserves the oriented graphs of  $\phi_1$  and  $\phi_2$ .

By Lemma 2 b) there exists  $x_1, x_2 \in \mathbb{R}$  such that

$$g_1^q(x_1) = x_1 + p, \quad g_2^q(x_2) = x_2 + p.$$

Without lost of generality suppose  $x_2 > x_1$ .

We denote

$$M_1 = (x_1, \phi_1(x_1)), M_2 = (x_2, \phi_2(x_2)), M_3 = (x_1, \phi_2(x_1)), M_4 = (x_2, \phi_1(x_1)).$$

As

$$F^q(x, \phi_1(x)) = (g^q(x), \phi_1(g^q(x))) = (x + p, \phi_1(x + p)) = (x + p, \phi_1(x)),$$

then  $M'_1 = G(M_1) = G(x_1, \phi_1(x)) = (x_1, \phi_1(x)) = M_1$ .

In the same way  $M'_2 = M_2$

By the twist condition,  $M'_3$  follows  $M_3$  in the orientation of the graph of  $\phi_2$  (see figure 4). As  $M_2$  is fixed, then  $M'_3$  precedes in order to  $M_2$ . That is,  $M'_3$  is interior to the arc (which is part of the graph of  $\phi_2$ ) connecting  $M_3$  and  $M_2$  (in this order as shown in figure 4). In the same way,  $M'_4$  is interior



to the arc (which is part of the graph of  $\phi_1$ ) connecting  $M_1$  and  $M_4$  (in this order).

Furthermore, by the twist condition, the image of the linear interval  $L_1 = [M_1, M_3]$  by  $G$  is situated to the right of the line defined by  $M_1$  and  $M_3$ . By the same reason, the image of the linear interval  $L_2 = [M_4, M_2]$  by  $G$  is situated to the left of the line defined by  $M_4$  and  $M_2$ .

Finally, as the graphs of  $\phi_1$  and  $\phi_2$  are invariant by  $G$ , then it follows from above that the region  $A$  delimited by them (and the lines  $M_1, M_3$  and  $M_2, M_4$ ) is mapped strictly inside  $A$  by the transformation  $G$ .

This is a contradiction with the fact that  $G$  preserves area. □

**Corollary 7.** *Let  $\phi_t : \mathbb{R} \rightarrow \mathbb{R}$ ,  $t \in [a, b]$ , be a family of continuous periodic functions of period 1. Suppose that  $\phi_t(x)$  is continuous as a function of  $(t, x)$  and that, for each  $t$ , the transformation  $\phi_t$  defines a rotational invariant circle  $\Gamma_t$ , such that they are disjoint two by two.*

*Then there exists a continuous strictly monotone function  $r : [a, b] \rightarrow \mathbb{R}$  such that*

$$\rho(\Gamma_t) = r(t) \pmod{1}, \quad \forall t \in [a, b].$$

**Proof:** By hypothesis, for each  $x \in \mathbb{R}$ , the function  $t \rightarrow \phi_t(x)$  is injective. Therefore, it is either strictly increasing or strictly decreasing. If it is increasing for some point  $x$ , it can not be decreasing for other points.

Without loss of generality, suppose that  $t \rightarrow \phi_t(x)$  is strictly increasing for all  $x \in \mathbb{R}$ .

Denote  $g_t(x) = F_1(x, \phi_t(x))$ . Then, by the claim after Definition 5, for all  $t \in [a, b]$ ,

$$\rho(\Gamma_t) = r(g_t) \pmod{1}.$$

Therefore, take  $r(t) = r(g_t)$ . Then,  $r$  is continuous (see [2] [9]) and strictly increasing (by Theorem 6). □

Consider now the torus  $Y = S^1 \times S^1$  with the natural orientation and let  $\pi_1, \pi_2 : Y \rightarrow S^1$  be the canonical projections. Denote by  $v$  the unitary vector field which is tangent to the fibers of  $\pi_1$  (compatible with the orientation of  $S^1$ ).

Consider a  $C^1$  diffeomorphism  $f : Y \rightarrow Y$  such that

- a)  $f$  preserves orientation of  $Y$ ,
- b)  $f$  preserves area of a Riemannian metric on  $Y$ ,

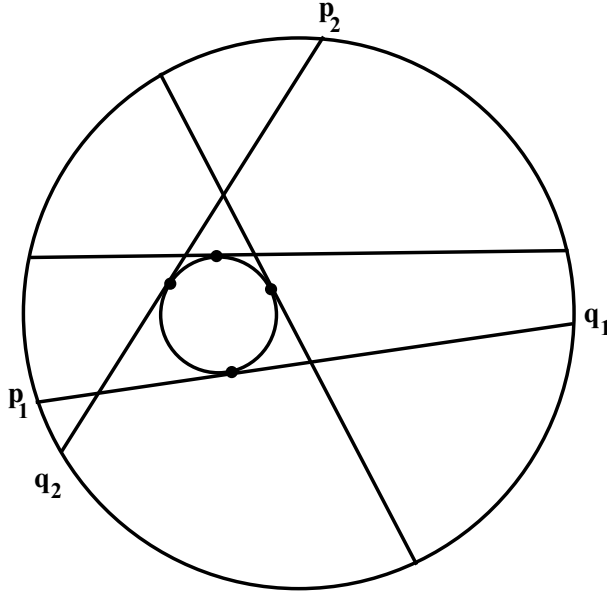


Figure 3:

c)  $v, df(v)$  are independent in each point and they are compatible (in this order) with the orientation of  $Y$  (twist condition).

**Definition 8.** Consider  $\phi : S^1 \rightarrow S^1$ , a continuous function and the oriented curve  $\Gamma \subset X$ , with parametrization  $x \rightarrow \pi(x, \phi(x))$ ,  $x \in S^1$ . We say  $\Gamma$  is an invariant rotational circle for  $f$ , if  $f(\Gamma) = \Gamma$  and  $f|_{\Gamma}$  preserves orientation.

We point out that  $\rho(f|_{\Gamma})$  is well defined and we denote  $\rho(\Gamma) = \rho(f|_{\Gamma})$ .

**Corollary 9.** Let  $\phi_t : S^1 \rightarrow S^1$ ,  $t \in [a, b]$ , be a family of continuous functions. Suppose that  $\phi_t(x)$  is continuous as a function of  $(t, x)$  and that for each  $t$  the transformation  $\phi_t$  defines a rotational invariant circle  $\Gamma_t$ , such that they are disjoint two by two.

Then there exists a continuous strictly monotone function  $r : [a, b] \rightarrow \mathbb{R}$  such that

$$\rho(\Gamma_t) = r(t) \pmod{1}, \quad \forall t \in [a, b].$$

**Proof:** Denote  $Y_0 = Y - \Gamma_b$ , then there exists a diffeomorphism  $\theta : X \rightarrow Y_0$  which preserves orientation, compatible with the projections  $\pi_1 : X \rightarrow S^1$ ,

$\pi_1|_{Y_0} : Y_0 \rightarrow S^1$ , and such that preserves the orientation of the fibers of these projections.

The transformation  $f_0 = \theta^{-1} \circ f \circ \theta : X \rightarrow X$  is a diffeomorphism of the cylinder  $X$  and which satisfies the properties of  $f : X \rightarrow X$  described just after Definition 4 ( $f_0$  preserves the ends of  $X$  because it preserves the orientation of  $Y$  and the orientation of  $\Gamma_b$ ).

For each  $t \in [a, b]$ , we have that  $\theta^{-1}(\Gamma_t)$  is a rotational invariant circle of  $f_0$ . In the same way as in Corollary 7, there exists  $r[a, b] \rightarrow \mathbb{R}$ , which is continuous, strictly monotone and such that

$$\rho(\Gamma_t) = r(t) \pmod{1}, \quad t \in [a, b].$$

Note that  $\pi_1|_{\Gamma_t} : \Gamma_t \rightarrow S^1$  is a diffeomorphism which preserves orientation.

By conjugacy,

$$\rho(\Gamma_t) = \rho((\pi_1|_{\Gamma_t}) \circ (f|_{\Gamma_t}) \circ (\pi_1|_{\Gamma_t})^{-1}).$$

Then, by Lemma 3 b), we have that

$$\lim_{t \rightarrow b^-} \rho(\Gamma_t) = \rho(\Gamma_b).$$

From this follows that

$$\lim_{t \rightarrow b^-} r(t) = l,$$

is finite. We define  $r(b) = l$  and

$$\rho(\Gamma_b) = r(b) \pmod{1}.$$

□

### 3 The main result

Using the previous notation for the Poncelet billiard, denote by  $Y$  the torus

$$Y = \{(A, r) : A \in K, r \text{ a line by } A\} = K \times S^1$$

The transformation  $f : Y \rightarrow Y$  associated to the Poncelet Billiard was defined before in section 1.  $f$  satisfies the hypothesis described in section 2.

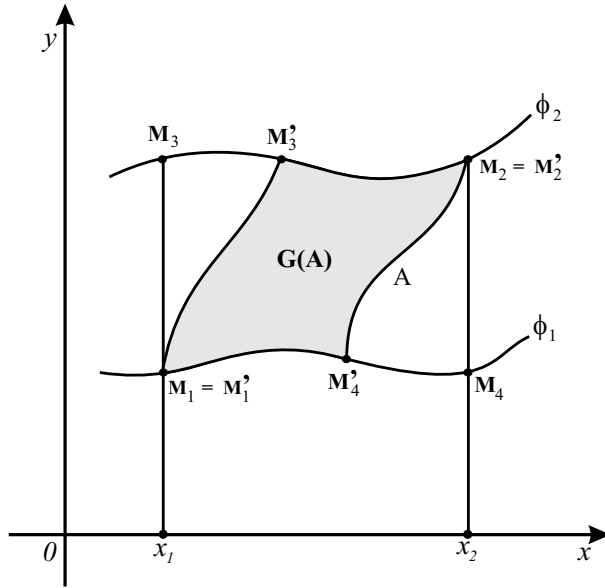


Figure 4:

Consider a fixed circle  $L$  of center  $0$  and radius  $t$ , where  $t \in [0, R - c]$ . To each point  $A \in K = S^1$ , one can associate a tangent  $r$  to  $L$  passing by  $A$ , and which has the property that  $L$  is on the left of  $r$  when oriented from  $A$  to  $A'$ .

In figure 3 we show the set of tangents to  $L$  defining the corresponding rotational invariant circle  $\Gamma_t$ .

For a fixed circle  $L$ , we define in such way a transformation  $\phi_t : S^1 \rightarrow S^1$ , which depends continuously on  $t$ . These  $\phi_t$  define a family of rotational invariant circles  $\Gamma_t$ , which are, two by two, disjoint.

By corollary 9, there exists  $r : [0, R - c] \rightarrow \mathbb{R}$ , continuous, strictly monotonous and such that

$$\rho(\Gamma_t) = r(t) \pmod{1}.$$

As  $f|_{\Gamma_{R-c}}$  has a fixed point, then  $\rho(\Gamma_{R-c}) = 0 \pmod{1}$ .

As a continuous limit of the figure 1 when  $t \rightarrow 0$ , one can see that  $f|_{\Gamma_0}$  has period 2, and therefore  $\rho(\Gamma_0) = 1/2 \pmod{1}$ .

Therefore,  $r(0) = \frac{1}{2} + n$ , and  $r(R - c) = m$ , where  $n, m \in \mathbb{Z}$ .

We can suppose that  $r(t)$  is such that  $r(0) = 1/2$ , and  $r(R - c) = k \in \mathbb{Z}$ .

We point out that  $f|_{\Gamma_t}$  has no fixed points, if  $t \neq (R - c)$ . Then,  $r(t)$  is not in  $\mathbb{Z}$ , if,  $t \neq (R - c)$  (Lemma 2 b). The conclusion is that  $k = 0$  or  $k = 1$ . Therefore,  $r$  is continuous, strictly increasing with values in  $[1/2, 1]$  or strictly decreasing with values in  $[0, 1/2]$ .

For each  $q = 3, 4, 5, \dots$ , denote  $e(q)$  the number of natural numbers  $n < q$ , which are relatively prime with  $q$ . Therefore, in any case, for a fixed  $q$ , there exactly  $\frac{e(q)}{2}$  numbers  $n$  in such way  $\frac{n}{q}$  is attained in the image of  $r$ .

From this and Lemma 2 b), it follows:

**Lemma 10.** *For each natural number  $q = 1, 2, 3, 4, \dots$ , there are  $\frac{e(q)}{2}$  rotational invariant circles  $\Gamma_t$ , such that  $f|_{\Gamma_t}$  has a periodic orbit of period exactly  $q$ .*

By Poncelet theorem [6] and [7],  $f|_{\Gamma_t}$  has a periodic orbit of period  $q$ , if and only if,  $f|_{\Gamma_t}$  is a  $q$ -periodic transformation. In this case the corresponding pair  $K, L$  is called a Poncelet pair.

From this follows theorem 1.

## 4 A second order estimate of the derivative of twist maps

The results of this sections are for a more general class of twist maps.

Consider a family  $g_t : \mathbb{R} \rightarrow \mathbb{R}$ , where  $a \leq t \leq b$ , of monotonous increasing homeomorphisms, such that

$$g_t(x + 1) = g_t(x) + 1, \quad x \in \mathbb{R}, \quad a \leq t \leq b.$$

From the identification  $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$ , from the family  $g_t$ , we obtain another one  $f_t : S^1 \rightarrow S^1$ , where  $a \leq t \leq b$ , of homeomorphisms of the circle which preserve orientation.

We know that is well defined

$$r(t) = \lim_{k \rightarrow \infty} \frac{g_t^k(x) - x}{k}, \quad a \leq t \leq b,$$

and the value  $r(t)$  indeed is independent of  $x$ .

We are interested in properties of the function  $r(t)$  and of the rotation number function  $\rho(t) = r(t) \pmod{1} \in \frac{\mathbb{R}}{\mathbb{Z}}$ .

We assume just the **twist condition**: there exists  $\frac{\partial g_t(x)}{\partial t}$  and is continuous and positive for all  $x \in \mathbb{R}$ ,  $t \in [a, b]$ .

As  $\frac{\partial g_t(x)}{\partial t}$  is periodic in  $x$ , we get

$$m = \inf_{a \leq t \leq b, x \in \mathbb{R}} \frac{\partial g_t(x)}{\partial t} > 0.$$

For a given irrational value  $x$  consider its development in continuous fraction and we call approximations of  $x$  the successive truncations of this infinite expansion.

We denote by defect approximation one which is smaller than  $x$  and by excess approximation one which is larger than  $x$ .

We refer the reader to [DK] for general properties of continuous fraction expansion.

We denote by  $F$  the sum of the inverses of the Fibonacci sequence:

$$F = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \dots$$

Consider two monotonous increasing homeomorphisms  $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$g_1(x+1) = g_1(x) + 1, \quad g_2(x+1) = g_2(x) + 1, \quad x \in \mathbb{R}.$$

Denote

$$r_1 = \lim_{k \rightarrow \infty} \frac{g_1^k(x) - x}{k}, \quad r_2 = \lim_{k \rightarrow \infty} \frac{g_2^k(x) - x}{k}.$$

Suppose  $g_1(x) < g_2(x)$ ,  $\forall x \in \mathbb{R}$ . As  $g_1(x) - g_2(x)$  is periodic and continuous, we have that

$$\alpha = \inf_{x \in \mathbb{R}} (g_2(x) - g_1(x)) > 0.$$

**Lemma 1.**

a)  $r_1 \leq r_2$ ,

b) Suppose that  $r_1$  is irrational and denote by  $\frac{p}{q}$  an excess continued fraction approximation of  $r_1$  (that is,  $\frac{p}{q} \geq r_1$ ) such that  $q > \frac{1}{\alpha}$ . Then,

$$r_1 < \frac{p}{q} \leq r_2.$$

c) suppose  $r_2$  is irrational and denote by  $\frac{p'}{q'}$  an excess continued fraction approximation of  $r_1$  (that is,  $\frac{p'}{q'} \geq r_1$ ) such that  $q' > \frac{1}{\alpha}$ . Then,

$$r_1 \leq \frac{p'}{q'} < r_2.$$

For a proof of the lemma see section 11.1 in [KH]

We need some results about approximation by truncation of continued fractions.

**Lemma 2.** For each irrational  $x$  consider  $\frac{p_n(x)}{q_n(x)}$ ,  $n = 1, 2, 3, \dots$ , the  $n$ -truncation of the continuous fractional expansion of  $x$ . Given  $0 < \epsilon < 1$ , then for Lebesgue almost every irrational  $x \in \mathbb{R}$ , there exists a sequence  $n_1 < n_2 < n_3, \dots$  of natural numbers (which depend on  $x$  and  $\epsilon$ ) such that

$$\frac{2e^{-2F}}{1+\epsilon} < \frac{q_{n_k+1}(x)}{q_{n_k}(x)} < \frac{2e^{2F}}{1-\epsilon}, \quad k = 1, 2, 3, \dots$$

**Proof:** We can suppose  $x \in (0, 1)$ . Denote  $T : [0, 1) \rightarrow [0, 1)$ , the Gauss transformation  $T(x) = \frac{1}{x} - [\frac{1}{x}]$ , if  $x \neq 0$ , and  $T(0) = 0$ .

For each irrational number  $x$ , from section 3.5 in [DK]

$$-\log q_n(x) = \log(x) + \log(T(x)) + \dots + \log(T^{n-1}(x)) + R(n, x), \quad n = 1, 2, 3, \dots,$$

where  $|R(n, x)| \leq F$ , for all  $x$  and  $n = 1, 2, 3, \dots$

Therefore,

$$\log\left(\frac{q_{n+1}(x)}{q_n(x)}\right) = -\log T^n(x) + R(n, x) - R(n+1, x),$$

and finally

$$-2F - \log T^n(x) \leq \log\left(\frac{q_{n+1}(x)}{q_n(x)}\right) \leq 2F - \log T^n(x).$$

As  $T$  is ergodic [DK], for Lebesgue almost every irrational  $x \in (0, 1)$ , we have that  $\frac{1}{2}$ , is a limit of a certain subsequence of  $T^n(x)$ . Therefore, there exists natural numbers  $n_1 < n_2 < n_3 < \dots$ , such that

$$\frac{1}{2} - \frac{\epsilon}{2} < T^{n_k}(x) < \frac{1}{2} + \frac{\epsilon}{2}, \quad k = 1, 2, 3, \dots$$

From this we get

$$-2F - \log\left(\frac{1}{2} + \frac{\epsilon}{2}\right) < \log\left(\frac{q_{n_k+1}(x)}{q_{n_k}(x)}\right) < 2F - \log\left(\frac{1}{2} - \frac{\epsilon}{2}\right).$$

□

**Corollary 1:** Given  $0 < \epsilon < 1$ , for almost irrational  $x$  there exists an infinite number of excess approximations  $\frac{p}{q}$ , and infinite number of excessive approximations  $\frac{p'}{q'}$ , such that

$$\frac{2e^{-2F}}{1+\epsilon} < \frac{q'}{q} < \frac{2e^{2F}}{1-\epsilon}, \text{ or } \frac{2e^{-2F}}{1+\epsilon} < \frac{q}{q'} < \frac{2e^{2F}}{1-\epsilon}.$$

**Proof:** In the Lemma 2 above take

$$p = p_{n_k}(x), q = q_{n_k}(x), p' = p_{n_k+1}(x), q' = q_{n_k+1}(x),$$

if  $n_k$  is odd and the opposite if  $n_k$  is even.

□

**Corollary 2:** Given  $0 < \epsilon < 1$ , denote

$$K_\epsilon = \frac{1-\epsilon}{e^{2F}(1+(1+\epsilon)e^{2F})^2}.$$

Then, for almost every irrational  $x \in \mathbb{R}$  there exists infinite excess approximations  $\frac{p}{q}$  and infinite defect approximations  $\frac{p'}{q'}$  of  $x$  such that:

$$\frac{p}{q} - \frac{p'}{q'} \geq K_\epsilon \left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

**Proof:**

Denote

$$L = \frac{e^{-2F}}{1+\epsilon}, M = \frac{e^{2F}}{1-\epsilon}.$$

From Corollary 1 we can suppose that:

a) first case:  $L < \frac{q'}{q} < M$

or,

b) second case  $L < \frac{q}{q'} < M$ .

In the first case:

$$\left(\frac{1}{q} + \frac{1}{q'}\right)^2 < \left(\frac{1}{q} + \frac{1}{Lq}\right)^2 = \frac{1}{q^2} \left(1 + \frac{1}{L}\right)^2$$



and,

$$\frac{p}{q} - \frac{p'}{q'} = \frac{pq' - qp'}{qq'} \geq \frac{1}{qq'} \geq \frac{1}{Mq^2}.$$

Therefore,

$$\frac{p}{q} - \frac{p'}{q'} > \frac{1}{M} \frac{1}{(1 + \frac{1}{L})^2} \left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

The claim follows from the fact that

$$K_\epsilon = \frac{1}{M} \frac{1}{(1 + \frac{1}{L})^2}.$$

The second case follows in a similar way. □

Now we will apply the results above for the function  $r(t)$ .

**Proposition 1:** The function  $r : [a, b] \rightarrow \mathbb{R}$  is continuous monotonous increasing. Moreover, if  $a \leq t_1 < t_2 \leq b$  and  $r(t_1) \notin \mathbb{Q}$  or  $r(t_2) \notin \mathbb{Q}$ , then  $r(t_1) < r(t_2)$ .

**Proof:** For the continuity and other properties see [KH]. The rest follows from Lemma 1. □

**Lemma 3:** If  $a \leq t_1 < t_2 \leq b$ , then  $g_{t_2}(x) - g_{t_1}(x) \geq m(t_2 - t_1)$ .

**Proof:** It follows easily from the twist condition and the mean value theorem. □

**Proposition 2:** There exist a set of full measure  $X \subset \mathbb{R}$  (that is,  $\mathbb{R} - X$  has Lebesgue measure zero) and such that if  $\tau \in (a, b)$  and  $r(\tau) \in X$ , then

$$\limsup_{t_2 \rightarrow \tau^-, t_1 \rightarrow \tau^+} \frac{r(t_2) - r(t_1)}{(t_2 - t_1)^2} \geq \frac{m^2}{e^{2F}(1 + e^{2F})^2}.$$

**Proof:** Given  $0 < \epsilon < 1$ , by corollary 2 there exists a set of full measure  $X_\epsilon \subset \mathbb{R}$  of irrational numbers such that if  $r(\tau) \in X_\epsilon$ , then the claim of this corollary is true for  $K_\epsilon$ . Let  $\delta$  be such that  $(\tau - \delta, \tau + \delta) \subset (a, b)$ .

The function  $g_\tau(x) - g_{\tau-\delta}(x)$  is continuous, periodic and positive by Lemma 3. Therefore,

$$\inf_{x \in \mathbb{R}} (g_\tau(x) - g_{\tau-\delta}(x)) > 0.$$

In the same way,

$$\inf_{x \in \mathbb{R}} (g_{\tau+\delta}(x) - g_\tau(x)) > 0.$$

By corollary 2, there exists approximations by excess  $\frac{p}{q}$  and by defect  $\frac{p'}{q'}$  of  $r(\tau)$  such that

$$\frac{1}{q'} < \inf_{x \in \mathbb{R}} (g_\tau(x) - g_{\tau-\delta}(x)),$$

$$\frac{1}{q} < \inf_{x \in \mathbb{R}} (g_{\tau+\delta}(x) - g_\tau(x))$$

and

$$\left(\frac{p}{q} - \frac{p'}{q'}\right) > K_\epsilon \left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

As,

$$\inf_{x \in \mathbb{R}} (g_t(x) - g_\tau(x)) = \min_{0 \leq x \leq 1} (g_t(x) - g_\tau(x)),$$

there exist  $\tau_1, \tau_2$  such that  $\tau - \delta < \tau_1 < \tau < \tau_2 < \tau + \delta$  and

$$\inf_{x \in \mathbb{R}} (g_\tau(x) - g_{\tau_1}(x)) = \frac{1}{q'}, \text{ and } \inf_{x \in \mathbb{R}} (g_{\tau_2}(x) - g_\tau(x)) = \frac{1}{q}.$$

Now, from Lemma 3, if  $t_1 < \tau_1 < \tau < \tau_2 < t_2$ , then

$$\inf_{x \in \mathbb{R}} (g_\tau(x) - g_{t_1}(x)) > \frac{1}{q'}, \text{ and } \inf_{x \in \mathbb{R}} (g_{t_2}(x) - g_\tau(x)) > \frac{1}{q}.$$

By lemma 1,

$$r(t_1) \leq \frac{p'}{q'} < r(\tau) < \frac{p}{q} \leq r(t_2).$$

Therefore,

$$r(t_2) - r(t_1) \geq \frac{p}{q} - \frac{p'}{q'} > K_\epsilon \left(\frac{1}{q} + \frac{1}{q'}\right)^2$$

As  $r$  is continuous, we get

$$r(\tau_2) - r(\tau_1) \geq K_\epsilon \left(\frac{1}{q} + \frac{1}{q'}\right)^2.$$

Now, by Lemma 3

$$g_{\tau_2}(x) - g_\tau(x) \geq m(\tau_2 - \tau), \quad \forall x \in \mathbb{R}$$

and,

$$g_\tau(x) - g_{\tau_1}(x) \geq m(\tau - \tau_1), \quad \forall x \in \mathbb{R}.$$

It follows that

$$\frac{1}{q} \geq m(\tau_2 - \tau), \text{ and } \frac{1}{q'} \geq m(\tau - \tau_1).$$

Therefore,

$$r(\tau_2) - r(\tau_1) \geq K_\epsilon m^2 (\tau_2 - \tau_1)^2,$$

and then

$$\limsup_{t_1 \rightarrow \tau^-, t_2 \rightarrow \tau^+} \frac{r(t_2) - r(t_1)}{(t_2 - t_1)^2} \geq K_\epsilon m^2.$$

The claim of the proposition follows from taking

$$X = \bigcap_{n=2}^{\infty} X_{\frac{1}{n}}.$$

□

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