Selection of ground states in the zero temperature limit for a one-parameter family of potentials

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Abstract

For the subshift of finite type $\Sigma = \{0, 1, 2\}^{\mathbb{N}}$ we study the convergence and the selection at temperature zero of the Gibbs measure associated to a non-locally constant Hölder potential which admits exactly two maximizing ergodic measures. These measures are Dirac measures at two different fixed points and the potential is flatter at one of these two fixed points.

We prove that there always is convergence but not necessarily to the Dirac measure at the point where the potential is the flattest. This is contrary to what was expected in the light of the analogous problem in Aubry-Mather theory [1]. This is also contrary to the finite range case where the equilibrium state converges to the equi-barycentre of the two Dirac measures.

Moreover we emphasize the strange behavior of the Gibbs measure: the eigenmeasure selects one Dirac measure (at the point where the potential is the flattest) and the eigenfunction selects the other one (at the point where the potential is the sharpest).

Keywords: selection of measures, transfer operator, Gibbs measures, equilibrium state, ergodic optimization.

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1 Introduction

1.1 optimization and selection and statements of results

In this paper we are interested in studying the problem of selection for convergence of Gibbs measures at temperature zero. For a dynamical system (X, T), it is usually very difficult to describe all the orbits $x \in X, T(x), T^2(x), \ldots$. The idea of Ergodic Theory is thus to describe orbits for *almost all* points, where almost all means with respect to some T-invariant probability measure. Again, usual dynamical systems have a lot of invariant probabilities, and the question is to find a way to emphasize some of them.

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Fixing $A : X \to \mathbb{R}$, the *thermodynamic formalism* provides such a way: a measure μ is an equilibrium state for A if it satisfies

$$h_\mu(T) + \int A\, d\mu = \sup_{\nu\ T-inv} \left\{ h_\nu(T) + \int A\, d\nu \right\},$$

where $h_\nu(T)$ is the usual Kolmogorov entropy. The supremum is taken over the set of T-invariant probabilities.This theory was deeply inspired by statistical mechanics, where the quantity $h_\mu(T) + \int A\, d\mu$ is (up to a sign) the *free energy* per site of a one-dimensional crystal. It was developed during the 70's essentially by Bowen, Ruelle and Sinai. We remind that for uniformly hyperbolic dynamics and "regular" potentials (*eg*, Hölder continuous), there is existence and uniqueness of the equilibrium state (see *eg* [2, 19, 20]).

For the last 10 years, a growing number of people have been studying a new way to distinguish T-invariant probabilities: instead of considering measures which maximize the free energy one focuses on measures which maximize the integral of the potential A; namely an A-maximizing measure is a T-invariant probability measure μ such that

$$\int A\, d\mu = \max_{\nu\ T-inv} \left\{ \int A\, d\nu \right\}.$$

There is a relation between these two approaches: for $\beta > 0$, let μ_β denotes an equilibrium state for βA and $\mathcal{P}(\beta)$ denotes the pressure

$$h_{\mu_\beta} + \beta \int A d\mu_\beta.$$

Then, under weak assumptions[1], the graph of $\mathcal{P}(\beta)$ admits an asymptote[2] as β goes to $+\infty$ whose slope is $\sup \left\{ \int A d\nu \right\}$. Moreover, any accumulation point for the equilibrium state μ_β as β goes to $+\infty$ is an A-maximizing measure (see [7, 17]).

In statistical mechanics, the parameter β is the inverse of the temperature, A is the opposite of the energy and maximizing measures are called *ground states* . The term b in the asymptote is called the *residual entropy* (see [10] (Appendix B2) for a survey of the problem in statistical mechanics.).

Roughly speaking, when the system is frozen[3], the equilibrium states go to ground states. Then, the problem of selection deals with the study of this "limit" at temperature zero:

1. Does the/one equilibrium state μ_β converge as β goes to $+\infty$?

2. If yes, what distinguishes the limit among the A-maximizing measures ?

In this paper we want to focus on that second question: what are the mechanisms or the parameters involved in the selection of the limit (if it does exist)?

We work here with a full shift Σ over the alphabet $\{0, 1, 2\}$: points in Σ are sequences $x = (x_0, x_1, \ldots)$ with $x_i \in \{0, 1, 2\}$. We will consider the usual terminology and the usual

[1] *e.g.* X compact Hausdorff, A and T continuous, metric entropy u.s.c..

[2] Namely $\lim_{\beta\to+\infty} \mathcal{P}(\beta) - a\beta - b = 0$; $a = \sup \left\{ \int A d\nu \right\}$.

[3] *i.e.* when the temperature goes to zero, or equivalently, when β goes to $+\infty$

product topology in Σ (see *e.g.* [18], chapter 1). Hence, we recall that a cylinder $[X_0, \ldots X_k]$ is the set of points $x = (x_n)$ such that $x_i = X_i$ for every $i \in [\![0, k]\!] := \{1, \ldots, k\}$. We equip Σ with the distance between $x = (x_n)$ and $y = (y_n)$ defined by

$$d(x, y) = \frac{1}{2^{\min\{n,\ x_n \neq y_n\}}}.$$

This distance is compatible with the product topology. It is non-canonical, and $\frac{1}{2}$ could be exchanged by any other real number θ in $(0, 1)$. However, we emphasize that the value of θ does not influence the Thermodynamic formalism.

We recall that the dynamics is given by the shift $\sigma : (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, \ldots)$. The two special points 0^∞ and 1^∞ respectively denote the points $(0, 0, \ldots)$ and $(1, 1, \ldots)$. They are fixed points for the shift σ over Σ.

We consider over this shift the Lipschitz potential A defined as follows:

$$A(x) = \begin{cases} -d(x, 0^\infty) & \text{if } x \in [0] \\ -3d(x, 1^\infty) & \text{if } x \in [1] \\ -\alpha & \text{otherwise} \end{cases}$$

for some $\alpha > 0$. This potential is always non-positive. There are only two maximizing measures, the two Dirac measures δ_{0^∞} and δ_{1^∞}, respectively at 0^∞ and 1^∞. Only these measures give zero integral for A. We point out that the potential is flatter[4] close to 0^∞.

The potential βA in Lipschitz hence admits a unique equilibrium state (for all $\beta \in \mathbb{R}$). It is a Gibbs measure (see also Subsection 1.2).

Our main result is:

Theorem *Let (Σ, σ) be the full 3-shift $(\{0, 1, 2\}^{\mathbb{N}}, \sigma)$ and A be the Hölder potential*

$$\begin{cases} -d(x, 0^\infty) & \textit{if } x \in [0] \\ -3d(x, 1^\infty) & \textit{if } x \in [1] \\ -\alpha & \textit{otherwise} \end{cases}$$

Let μ_β be the unique Gibbs measure associated to βA, $\beta \in \mathbb{R}$. Let ρ be the golden mean $\rho := \dfrac{1 + \sqrt{5}}{2}$. Then

1. *for $\alpha > 1$, μ_β converges to $\frac{1}{2}(\delta_{0^\infty} + \delta_{1^\infty})$ as β goes to $+\infty$,*
2. *for $\alpha = 1$, μ_β converges to $\frac{1}{1+\rho^2}(\rho^2 \delta_{0^\infty} + \delta_{1^\infty})$ as β goes to $+\infty$,*
3. *for $0 < \alpha < 1$, μ_β converges to δ_{0^∞} as β goes to $+\infty$.*

This potential was inspired by a result of selection for Hamiltonian/Lagrangian setting in [1]. Our principal motivation for this paper was to study if flatness of the potential plays a role in the selection as it does in [1].

In our mind the importance of the result does not rest on the values but on the diversity of the values. This clearly means that flatness is not a determinant argument in the selection, contrarily to what was expected.

[4] More precisely A is sharper close to 1^∞.

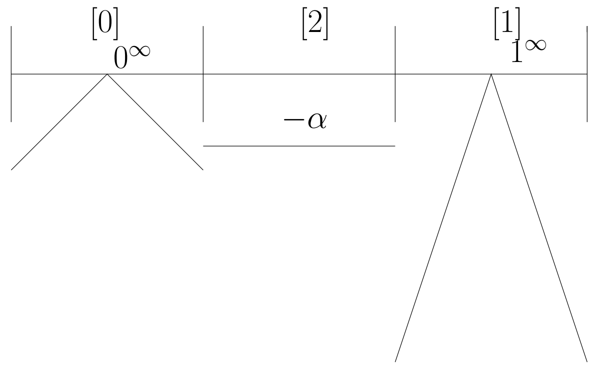

Figure 1: Potential A

Our result also shows that the theory of selection is extremely wild and, even in an apparently very simple situation with convergence, selection seems to be unpredictable. This completes the state of the art on that topic, and shows that a general theory is certainly far from being reachable.

We recall that generically for the $\mathcal{C}^0$ topology, there exists a unique maximizing measure. Therefore convergence occurs ! For a subshift of finite and if A is locally constant, then μ_β always converges (see [4, 13, 6]) and the selection is well identified (see [13, 6]). Contrarily to what could be expected, not every maximizing component with maximal entropy has positive limit measure, but only components with maximal entropy and which are the most "isolated". These are the vice-maximizing periodic orbits between clusters which determine the selection (see [13]). See also [12] (section 9) for an explicit computation. On the opposite way, examples of non-convergence are also known (see [11, 5]).

To emphasize, if still necessary, how wild selection is, we recall that, in the case of locally constant potential, if there are only two maximizing ergodic measures with maximal entropy, then μ_β converges to the middle of these two ergodic measures. In our case this does not occurs if $\alpha \leq 1$.

With our setting, μ_β is unique and is actually a *Gibbs measure*: there is an operator $\mathcal{L}_\beta$ naturally associated to the problem and μ_β is the product of specific eigenmeasure and eigenfunction of that operator (see [18] and Subsection 1.2 here for definitions). A very curious phenomena is that the eigenmeasure and the eigenfunction have opposite behavior. When the system is frozen, the eigenmeasure becomes exponentially bigger around 0^∞ than around 1^∞ (see Cor. 3.8). For the eigenfunction the opposite happens (see Prop. 2.4). In some sense, the eigenmeasure selects one maximizing measure and the eigenfunction selects the other one. As a consequence of this opposite behavior, the selection does not appear at the exponential scale (see below).

The presence of the Golden mean is (probably) not due to some "universal" constant, but is more surely an accident. Nevertheless we are not completely sure on where it exactly comes from. Ph. Thieullen pointed out it could be related to the fact that there are 2 maxima. We believe this is also related to the special values we chose (the "3" for A and the "2" for the distance). We can prove that the same kind of result holds if we replace $-3d(x, 1^\infty)$ with some $-\Gamma d(x, 1^\infty)$, with $\Gamma > 1$, and the $\dfrac{1}{2}$ in the distance by some $\theta \in (0, 1)$, but for the sake of compactness we do not include the proofs here. Indeed, the computation for more general case would be a little bit more complicated and the formulas less convenient to be used.

1.2 More notations - plan of the proof

If $y = (y_1, y_2, \ldots)$ is a point in Σ and if $a = 0, 1, 2$, we denote the point $(a, y_1, y_2, \ldots)$ in Σ by ay.

The main tool is the transfer operator (also called Ruelle-Perron-Frobenius operator) defined as follows:

$$\begin{aligned} \mathcal{L}_\beta\varphi(x) &= \sum_{y\in\sigma^{-1}(x)} e^{\beta A(y)}\varphi(y) \\ &= e^{-\beta d(0x,0^\infty)}\varphi(0x) + e^{-\beta d(1x,1^\infty)}\varphi(1x) + e^{-\alpha\beta}\varphi(2x). \end{aligned}$$

where β is the inverse of the temperature. We recall here some of its properties (see *e.g.* [2]). It acts on continuous functions and its dual operator, denoted by $\mathcal{L}^*_\beta$, acts on probability measures.

We know that there exists some function H_β and some probability measure ν_β such that $\mathcal{L}_\beta(H_\beta) = e^{P(\beta)}H_\beta$ and $\mathcal{L}^*_\beta(\nu_\beta) = e^{P(\beta)}\nu_\beta$. Then, the measure defined by $d\mu_\beta = H_\beta d\nu_\beta$ is $\sigma-$invariant and the unique equilibrium state associated to βA (if it is normalized to get a probability measure).

Throughout, they will be referred to as the eigenmeasure and the eigenfunction. Most of the time we will omit the subscript β.

The plan of the proof of the main result of the paper is the following:

In Section 2 we give the exponential asymptotics for the eigenfunction (obtaining what is called a calibrated subaction) and the pressure.

In Section 3 we prove the convergence of the eigenmeasure to δ_{0^∞}. For this we give precise values for the ν-measures of rings $[0^n] \setminus [0^{n+1}]$ and $[1^n] \setminus [1^{n+1}]$.

In Section 4 we compute the exact values of the eigenfunction on the same rings considered before in Section 3.

In Section 5 we finish the proof of our Theorem.

2 Exponential asymptotic for the pressure and the eigenfunction

We first recall some tools introduced to study Thermodynamic formalism at temperature zero (see *e.g.* [9]). In the following, $m(A)$ denotes $\sup\left\{\int A\,d\mu\right\}$ (which is zero in our case).

Definition 2.1. *We say that $u : \Sigma \to \mathbb{R}$ is a calibrated subaction for A if for any y in Σ we have*

$$u(y) = \sup_{\sigma(x)=y} \{A(x) + u(x) - m(A)\}.$$

The family of functions $\{\frac{1}{\beta}\log H_\beta\}_{\beta\in\mathbb{R}^+}$ is uniformly bounded and equi-continuous ; we denote by V any accumulation point for $\frac{1}{\beta}\log H_\beta$ as β goes to $+\infty$ (and for the $\mathcal{C}^0$-norm). It is a calibrated subaction, see [7]. We will exhibit a more explicit expression of such V.

We recall that the Peierls' barrier is given by

$$h(x,y)=\lim_{\epsilon\to 0}\sup_{n\geq 1}\left\{\sum_{j=0}^{n-1}\left(A(\sigma^j(z))-m(A)\right),\sigma^n(z)=y,d(z,x)<\epsilon\right\}.$$

Remark 1. We leave it to the reader to check that for every $x\neq 0^\infty,1^\infty$ both numbers $h(0^\infty,x)$ and $h(1^\infty,x)$ are negative.

Let $\Omega\subset\Sigma$ be the Aubry set of A (see [7] for a definition). Then, it is proved in [9] (see Theorem 10), that every calibrated subaction u satisfies

$$u(y)=\sup_{\mathbf{x}\in\Omega}[h(\mathbf{x},y)+u(\mathbf{x})], \tag{1}$$

In the present case the Aubry set is the union of the two fixed points $p=0^\infty$ and $q=1^\infty$. In this way, any calibrated subaction is determined by its values on p and q.

Lemma 2.2. *The functions defined by $u_0(x)=-d(x,0^\infty)$ and $u_1(x)=-3d(x,1^\infty)$ are both calibrated subactions.*

Proof. The proof is only done for u_0, the other case being similar. We consider $y\in\Sigma$ and we want to prove

$$-d(0^\infty,y)=:u_0(y)=max\{A(0y)+u_0(0y),A(1y)+u_0(1y),A(2y)+u_0(2y)\}. \tag{2}$$

We set $y=(y_0,y_1,y_2,\ldots)$. We first assume that $y_0\neq 0$. Note that both $A(1y)$ and $A(2y)$ are non-positive and $u_0(1y)=u_0(2y)=-1$. Hence $u_0(y)=-1$ is bigger than (or equal to) both terms $A(1y)+u_0(1y)$ and $A(2y)+u_0(2y)$. Now $A(0y)=-\frac{1}{2}$ and $u_0(y)=-\frac{1}{2}$. Hence (2) holds in that case. Assume now that y belong to the cylinder 0^n and $y_{n+1}\neq 0$. Then $u_0(y)=\frac{-1}{2^n}$. Again, note that $u_0(y)$ is bigger than both terms $A(1y)+u_0(1y)$ and $A(2y)+u_0(2y)$. We also get

$$\frac{-1}{2^n}=\frac{-1}{2^{n+1}}+\frac{-1}{2^{n+1}}=A(0y)+u_0(0y).$$

Hence, (2) holds in that case too. □

Using Lemma 2.2 we can get a more simple formulation for V.

Lemma 2.3.

$$V(x)=\sup\{[V(0^\infty)-d(0^\infty,x)],[V(1^\infty)-3\,d(1^\infty,x)]\}$$

Proof. As V is a calibrated subaction, Equation (1) holds with V instead of u. Then we claim that

$$h(0^\infty,y)=u_0(y)\text{ and }h(1^\infty,y)=u_1(y).$$

The Lemma follows from this claim and Equation (1).

Let us prove the claim. Again, we only prove that $h(0^\infty,x)=u_0(x)=-d(x,0^\infty)$, the other equality being similar.

Let $x=(x_0,x_1,\ldots)$ be in Σ. We get

$$u_0(x)=\max(h(0^\infty,x)+u_0(0^\infty),h(1^\infty,x)+u_0(1^\infty))=\max(h(0^\infty,x),h(1^\infty,x)-1).$$

Note that $u_0(x) \geq -1$ and by Remark 1 the Peierls barriers are both negative. Hence we obtain

$$u_0(x) = h(0^\infty, x).$$

$\square$

Now, we use properties of the eigenfunction H_β to obtain some relations satisfied by V. A calibrated subaction, in the present situation, is determined by its values 0^∞ and 1^∞. We just need the relative values of V at these points.

Proposition 2.4. *For $\alpha > 1$, we get $V(1^\infty) = V(0^\infty) + 1$ and $\lim_{\beta\to+\infty} \frac{1}{\beta} \log P(\beta) = -2$.*

For $0 < \alpha \leq 1$, we get $V(1^\infty) = V(0^\infty) + \alpha$ and $\lim_{\beta\to+\infty} \frac{1}{\beta} \log P(\beta) = -(1+\alpha)$.

Proof. Let γ be an accumulation point for $\frac{1}{\beta} \log P(\beta)$ as β goes to $+\infty$. For simplicity of notations we still write $\lim_{\beta\to+\infty} \frac{1}{\beta} \log$ even if we only consider a subsequence (β_n).

From the equation $\mathcal{L}_\beta(H_\beta) = e^{P(\beta)} H_\beta$ we get the pair of equations[5]

$$(e^{P(\beta)} - 1)\, H_\beta(0^\infty) = e^{-\alpha\beta} H_\beta(2) + e^{-\frac{3}{2}\beta}\, H_\beta(1\,0^\infty), \tag{3a}$$

$$(e^{P(\beta)} - 1)\, H_\beta(1^\infty) = e^{-\alpha\beta} H_\beta(2) + e^{-\frac{1}{2}\beta}\, H_\beta(0\,1^\infty). \tag{3b}$$

Remember that V is an accumulation point for $\frac{1}{\beta} \log H_\beta$ and by Lemma 2.2

$$\begin{aligned} V(10^\infty) &= \max\{[V(0^\infty) - 1], [V(1^\infty) - \frac{3}{2}]\}, \\ V(2\,x_1\,x_2..) &= \max\{[V(0^\infty) - 1], [V(1^\infty) - 1]\}, \\ V(01^\infty) &= \max\{[V(0^\infty) - \frac{1}{2}], [V(1^\infty) - 3\,]\}. \end{aligned}$$

Then, taking $\frac{1}{\beta}\log$ in Equation (3a) and making β go to $+\infty$ we get

$$\begin{aligned} \gamma + V(0^\infty) &= \max\{[V(0^\infty) - 1 - \alpha], [V(1^\infty) - 3 - \alpha], [V(0^\infty) - 1 - \frac{3}{2}], [V(1^\infty) - \frac{3}{2} - \frac{3}{2}]\} \\ &= \max\{[V(0^\infty) - 1 - \alpha], [V(1^\infty) - 3 - \alpha], [V(0^\infty) - \frac{5}{2}], [V(1^\infty) - 3]\} \\ &= \max\{[V(0^\infty) - 1 - \alpha], [V(0^\infty) - \frac{5}{2}], [V(1^\infty) - 3]\}. \end{aligned} \tag{4}$$

Similarly, from (3b) we finally get

$$\gamma + V(1^\infty) = \max\{[V(0^\infty) - 1], [V(1^\infty) - 7/2], [V(1^\infty) - 3 - \alpha]\}. \tag{5}$$

We first deal with the case $\alpha > 1$. We will show that $V(1^\infty) = V(0^\infty) + 1$. We divide the analysis in two cases:

[5] See Lemma 4.1 to check that H_β is constant on cylinder [2].

1) if $\alpha > 3/2$, then, we have to solve

$$\gamma + V(0^\infty) = \max\{[V(0^\infty) - \frac{5}{2}], [V(1^\infty) - 3]\}, \quad (6a)$$
$$\gamma + V(1^\infty) = \max\{[V(0^\infty) - 1], [V(1^\infty) - 7/2]\}. \quad (6b)$$

Now, we show that this system of equation is solvable if and only if $V(0^\infty) - \frac{5}{2} \leq V(1^\infty) - 3$ and $V(0^\infty) - 1 \geq V(1^\infty) - 7/2$.

Suppose that $V(0^\infty) - \frac{5}{2} > V(1^\infty) - 3$. Then, we get $\gamma + V(0^\infty) = V(0^\infty) - 5/2$, which shows that we have $\gamma = -5/2$. Thus, we must have $V(0^\infty) - 1 \geq V(1^\infty) - 7/2$ (otherwise (6b) would give $\gamma = -\frac{7}{2}$), and we get

$$V(0^\infty) - 1 = \gamma + V(1^\infty) = -5/2 + V(1^\infty).$$

From this follows that $V(1^\infty) = 3/2 + V(0^\infty)$. This yields

$$V(1^\infty) - 3 = (3/2 + V(0^\infty)) - 3 = V(0^\infty) - \frac{3}{2} > V(0^\infty) - \frac{5}{2},$$

which produces a contradiction.

Then, we have

$$\gamma + V(0^\infty) = V(1^\infty) - 3 \quad (7)$$

An important consequence is that we must get $\gamma \geq -\frac{5}{2}$. If $V(0^\infty) - 1 \leq V(1^\infty) - 7/2$, then (6b) shows that γ is equal to $-\frac{7}{2}$ which is impossible. Hence

$$\gamma + V(1^\infty) = V(0^\infty) - 1. \quad (8)$$

Finally, (7) and (8) yield $\gamma = -2$, and $V(1^\infty) = V(0^\infty) + 1$.

2) The case $1 < \alpha \leq \frac{3}{2}$. The proof is similar. It is explicitly reproduced here, but the reader can skip it in a first reading.

The new system to solve is

$$\gamma + V(0^\infty) = \max\{[V(0^\infty) - (1 + \alpha)], [V(1^\infty) - 3]\}, \quad (9a)$$
$$\gamma + V(1^\infty) = \max\{[V(0^\infty) - 1], [V(1^\infty) - 7/2]\}. \quad (9b)$$

Again, we show that this system of equation is solvable if, and only if, $V(0^\infty) - (1+\alpha) \leq V(1^\infty) - 3$ and $V(0^\infty) - 1 \geq V(1^\infty) - 7/2$.

Suppose that $V(0^\infty) - (1 + \alpha) > V(1^\infty) - 3$. Then, we get $\gamma + V(0^\infty) = V(0^\infty) - (1 + \alpha)$, which shows that we have $\gamma = -(1 + \alpha) > -\frac{5}{2}$. Thus, we must have $V(0^\infty) - 1 \geq V(1^\infty) - 7/2$ (otherwise (9b) would give $\gamma = -\frac{7}{2}$), and we get

$$V(0^\infty) - 1 = \gamma + V(1^\infty) = -(1 + \alpha) + V(1^\infty).$$

From this follows that $V(1^\infty) = \alpha + V(0^\infty)$. This yields

$$V(1^\infty) - 3 = (\alpha + V(0^\infty)) - 3 = V(0^\infty) - 2 > V(0^\infty) - \frac{5}{2},$$

which produces a contradiction.

Then, we have

$$\gamma + V(0^\infty) = V(1^\infty) - 3 \tag{10}$$

An important consequence is that $\gamma \geq -(1+\alpha) > -\frac{5}{2}$. If $V(0^\infty) - 1 \leq V(1^\infty) - 7/2$, then (9b) shows that γ is equal to $-\frac{7}{2}$ which is impossible. Hence

$$\gamma + V(1^\infty) = V(0^\infty) - 1. \tag{11}$$

Finally, (10) and (11) yield $\gamma = -2$, and $V(1^\infty) = V(0^\infty) + 1$.

We point out here that the above discussion can be done for every sub-family of β's. In particular, this shows that $\dfrac{1}{\beta}\log P(\beta)$ can have only one accumulation point. In other words, it converges to $\gamma = -2$.

Now, we deal with the case $\alpha \leq 1$. We will show that $V(1^\infty) = V(0^\infty) + \alpha$. The system we have to solve is

$$\begin{aligned} \gamma + V(0^\infty) &= \max\{[V(0^\infty) - (1+\alpha)], [V(1^\infty) - 3]\}, && (12a)\\ \gamma + V(1^\infty) &= \max\{[V(0^\infty) - 1], [V(1^\infty) - 7/2], [V(1^\infty) - 3 - \alpha]\}. && (12b) \end{aligned}$$

We show that, whatever is the case $\alpha \leq \frac{1}{2}$ or $\alpha \geq \frac{1}{2}$, the system can be solved if, and only if, $V(0^\infty) - (1+\alpha) \geq V(1^\infty) - 3$ and $V(0^\infty) - 1 \geq V(1^\infty) - 7/2, V(1^\infty) - 3 - \alpha$.

Let us proceed by contradiction and assume we get $V(0^\infty) - (1+\alpha) < V(1^\infty) - 3$. In that case, if we assume that we get $V(0^\infty) - 1 \geq V(1^\infty) - 7/2, V(1^\infty) - 3 - \alpha$, then the system to solve is exactly given by equations (7) and (8). This yields $\gamma = -2$, and $V(1^\infty) = V(0^\infty) + 1$.

Then, we get $V(1^\infty) - 3 = V(0^\infty) - 2 \leq V(0^\infty) - (1+\alpha)$ which produced a contradiction with our assumption $V(0^\infty) - (1+\alpha) < V(1^\infty) - 3$.

This means that $V(0^\infty) - 1 \leq V(1^\infty) - 7/2, V(1^\infty) - 3 - \alpha$, and the bigger term only depends on the relative position of α with respect to $\frac{1}{2}$. Depending of this position, we get $\gamma = -\dfrac{7}{2}$ or $\gamma = -3 - \alpha$. Then (12a) would give in both case

$$V(0^\infty) - \gamma > V(0^\infty) - (1+\alpha),$$

which produces a contradiction. Hence, we must get $V(0^\infty) - (1+\alpha) \geq V(1^\infty) - 3$ and

$$\gamma = -(1+\alpha). \tag{13}$$

If $V(0^\infty) - 1 \geq V(1^\infty) - 7/2, V(1^\infty) - 3 - \alpha$ does not hold, then we would get $\gamma = -\dfrac{7}{2}$ or $\gamma = -3 - \alpha$, which is impossible. Thus we must get $V(0^\infty) - 1 \geq V(1^\infty) - 7/2, V(1^\infty) - 3 - \alpha$ and we finally get

$$V(1^\infty) + \gamma = V(1^\infty) - (1+\alpha) = V(0^\infty) - 1. \tag{14}$$

This finishes the proof of the proposition (again γ is the unique possible accumulation point for $\dfrac{1}{\beta}\log P(\beta)$).

□

3 The eigenmeasure ν

In this section we study the eigenmeasure $\nu_{\beta A}$. We prove that it converges to the Dirac measure δ_{0^∞}. We also estimate the limit ratio of measures on rings of the form $[0^n]\backslash[0^{n+1}]$ and $[1^n]\backslash[1^{n+1}]$.

3.1 A useful function

We define and study a function F depending on the pressure $P(\beta)$ and on the parameter β.

Definition 3.1. *For $Z \geq 0$ and $\beta \geq 0$ $F(Z,\beta) := \sum_{k=0}^{\infty} e^{-kZ} e^{\frac{\beta}{2^{k+1}}}$ and its partial sums $F_n(Z,\beta) := \sum_{k=0}^{n} e^{-kZ} e^{\frac{\beta}{2^{k+1}}}$.*

Clearly, $F_n(Z,\beta) \to F(Z,\beta)$ when $n \to \infty$.

We recall that as β goes to $+\infty$, P goes exponentially fast to 0. The asymptotic behavior of F (for β very large) can be obtained as follows:

Lemma 3.2. *For every $\beta > 2\ln 2$ we get*

$$\left| F(P,\beta) - \frac{1}{P} \right| \leq \frac{\beta e^{\beta/2}}{2\,\ln 2}(2 + \sum_{n\geq 1} (\frac{P}{\ln 2})^n).$$

Proof. Let us consider a positive Z. Note that the function $x \mapsto -Zx + \frac{\beta}{2.2^x}$ is decreasing on $\mathbb{R}_+$. We can thus compare the sum and the integral:

$$\int_0^{+\infty} Z e^{-xZ} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx \leq ZF(Z,\beta) \leq \int_0^{+\infty} Z e^{-xZ} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx + Ze^{\frac{\beta}{2}}.$$

Let us study the integral. We get

$$\begin{aligned}\int_0^{+\infty} Z e^{-xZ} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx &= \left[-e^{-xZ} e^{\frac{\beta}{2}\frac{1}{2^x}}\right]_0^{+\infty} - \int_0^{+\infty} \frac{\beta}{2} e^{-xZ} \frac{\ln 2}{2^x} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx. \\ &= e^{\frac{\beta}{2}} - \int_0^{+\infty} \frac{\beta}{2} e^{-xZ} \frac{\ln 2}{2^x} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx.\end{aligned}$$

Let us set $u = \frac{1}{2^x}$ in this last integral. We get

$$\int_0^{+\infty} Z e^{-xZ} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx = e^{\frac{\beta}{2}} - \int_0^1 \frac{\beta}{2} e^{-Z\frac{\ln u}{\ln 2}} e^{\frac{\beta}{2}u}\,du.$$

Writing $e^{-Z\frac{\ln u}{\ln 2}} = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(-Z\frac{\ln u}{\ln 2}\right)^n$ we get

$$\int_0^{+\infty} Z e^{-xZ} e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx = e^{\frac{\beta}{2}} - \int_0^1 \frac{\beta}{2} \sum_{n=0}^{+\infty} \frac{1}{n!}\left(-Z\frac{\ln u}{\ln 2}\right)^n e^{\frac{\beta}{2}u}\,du.$$

To get the inverse of the two sums we remind that $\int_0^1 |\ln u|^n\,du = \int_0^{+\infty} v^n e^{-v}\,dv = n!$. Then for $Z < \ln 2$ we get

$$\begin{aligned}\int_0^{+\infty} Ze^{-xZ}e^{\frac{\beta}{2}\frac{1}{2^x}}\,dx &= e^{\frac{\beta}{2}} - \sum_{n=0}^{+\infty}\frac{1}{n!}\left(\frac{-Z}{\ln 2}\right)^n \int_0^1 \frac{\beta}{2}(\ln u)^n e^{\frac{\beta}{2}u}\,du \\ &= 1 - \sum_{n=1}^{+\infty}\frac{1}{n!}\left(\frac{-Z}{\ln 2}\right)^n \int_0^1 \frac{\beta}{2}(\ln u)^n e^{\frac{\beta}{2}u}\,du.\end{aligned}$$

Now, note that

$$\left|\frac{1}{n!}\left(\frac{-Z}{\ln 2}\right)^n \int_0^1 \frac{\beta}{2}(\ln u)^n e^{\frac{\beta}{2}u}\,du\right| \leq \frac{1}{n!}\left(\frac{Z}{\ln 2}\right)^n \frac{\beta}{2}e^{\frac{\beta}{2}} \int_0^1 |\ln u|^n\,du = \left(\frac{Z}{\ln 2}\right)^n \frac{\beta}{2}e^{\frac{\beta}{2}}.$$

We also recall that for positive β, the pressure is strictly smaller than the topological entropy $\ln 3$. This shows the lemma. □

3.2 The eigenmeasure on the cylinders [0] and [1]

We remind that the eigen-probability for βA, ν_β, is a conformal measure: for any cylinder set B

$$\nu_\beta(\sigma(B)) = \int_B e^{P(\beta)-\beta A(x)}d\,\nu_\beta(x).$$

We shall use this simple relation to compute exact values for ν_β of some special cylinders.

For simplicity we drop the subscribe β in ν_β and simply write ν. We shall also use the notation $*_0$ for the pair of symbols which are not 0 and $*_1$ for the pair of symbols which are not 1. Then

$$[0*_0] = [01] \sqcup [02] \qquad \text{and} \qquad [1*_1] = [10] \sqcup [12]$$

(and the unions are disjoint).

We can now estimate the measures of the cylinders [0] and [1].

Lemma 3.3.

$$\nu[0] = e^{-\frac{\beta}{2}}\,F(P,\beta)\,\nu[0*_0]$$

$$\nu[1] = e^{-\frac{3\beta}{2}}\,F(P,3\beta)\,\nu[1*_1]$$

Proof. Conformality yields

$$\nu[0*_0] = \nu[\sigma(00*_0)] = e^{P+\frac{\beta}{2^2}}\,\nu[00*_0] = e^{2P+\frac{\beta}{2^3}+\frac{\beta}{2^3}}\,\nu[000*_0],$$

and so on. By induction we get

$$\nu[0*_0] = e^{(n-1)\,P+\beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}\,\nu[\underbrace{00\ldots0}_{n}*_0]. \tag{15}$$

Hence, we get

$$\nu[0] = \sum_{n=1}^{\infty}\nu[\underbrace{00\ldots0}_{n}*_0] = \sum_{n=1}^{\infty} e^{-(n-1)\,P}\;e^{-\frac{\beta}{2}}\,e^{\frac{\beta}{2^n}}\,\nu[0*_0] = \;e^{-\frac{\beta}{2}}\,F(P,\beta)\,\nu[0*_0].$$

Similarly we get $\nu[1] = \;e^{-\frac{3\,\beta}{2}}\,F(P,3\,\beta)\,\nu[1*_1]$. □

Using $[0*_0]=[01]\sqcup[02]$ and$[1*_1]=[10]\sqcup[12]$ and the conformal property of ν we obtain the following system:

$$\nu[1*_1] = \nu[2]\,e^{-P-\frac{3\beta}{2}}+\nu[0]e^{-P-\frac{3\beta}{2}}. \tag{16a}$$
$$\nu[0*_0] = \nu[2]\,e^{-P-\frac{\beta}{2}}+\nu[1]e^{-P-\frac{\beta}{2}}. \tag{16b}$$

This system is the key point to determine the convergence of the eigenmeasure.

Proposition 3.4. *The ratio* $\dfrac{\nu[0]}{\nu[1]}$ *goes exponentially fast to* $+\infty$ *as* β *goes to* $+\infty$*.*

Proof. By Lemma 3.3 the system (16) can be transformed into a system in $\nu[0]$, $\nu[1]$, and $\nu[2]$:

$$\begin{aligned}\nu[0] &= e^{-\beta/2}\,F(P,\beta)\,\{\nu[2]\,e^{-P-\frac{\beta}{2}}+\nu[1]e^{-P-\frac{\beta}{2}}\,\}\\ \nu[1] &= e^{-(3\,\beta)/2}\,F(P,3\beta)\,\{\nu[2]\,e^{-P-\frac{3\beta}{2}}+\nu[0]e^{-P-\frac{3\beta}{2}}\,\}\end{aligned}$$

This yields

$$\frac{\nu[0]}{\nu[1]}=e^{2\beta}\,\frac{F(P,\beta)\,(\,1+e^{-P-3\,\beta}\,F(P,3\beta)\,)}{F(P,3\beta)\,(\,1+e^{-P-\beta}\,F(P,\beta)\,)} \tag{17}$$

Finally, when $\beta\to\infty$, $\dfrac{\nu[0]}{\nu[1]}$ goes to $+\infty$ exponentially fast: roughly speaking, Proposition 2.4 and Lemma 3.2 show that $e^{-P-3\,\beta}\,F(P,3\beta)$ behaves as $e^{-\beta}$, $(1+e^{-P-\beta}\,F(P,\beta))$ behaves as e^{β}, and $\dfrac{F(P,\beta)}{F(P,3\beta)}$ behaves as 1.

Considering the terms of higher orders given by Lemma 3.2, $\dfrac{\nu[0]}{\nu[1]}$ goes to $+\infty$ faster than any $e^{(1-\varepsilon)\beta}$ for any positive ε.

□

We point out that Lemma 3.3 also allows to transform the system (16) into a system in $\nu([0*_0])$, $\nu([1*_1])$, and $\nu(2)$. From this system we get

$$\frac{\nu[0*_0]}{\nu[1*_1]}=e^{\,\beta}\,\frac{(\,1+e^{-P-3\beta}\,F(P,3\,\beta)\,)}{(\,1+e^{-P-\beta}\,F(P,\beta)\,)}. \tag{18}$$

Nevertheless, at this point of the proof we do not have enough information on P to compute the limit of the ratio. Proposition 2.4 and Lemma 3.2 just ensure that $\dfrac{1}{\beta}\log\dfrac{\nu[0*_0]}{\nu[1*_1]}$ goes to 0. However, we can get ratios for other rings:

Corollary 3.5. *For every* $n\geq 2$,

$$\frac{\nu[0^n*_0]}{\nu[1^n*_1]}=e^{\beta(1-\frac{1}{2^{n-1}})}\,\frac{\nu[0*_0]}{\nu[1*_1]}.$$

For every positive ε*, the ratio* $\dfrac{\nu[0^n*_0]}{\nu[1^n*_1]}$ *goes to* $+\infty$ *as* β *goes to* $+\infty$ *faster than* $e^{\beta(1-\frac{1}{2^{n-1}}-\varepsilon)}$.

3.3 Convergence of the eigenmeasure

In this subsection we get a finer estimate for $P(\beta)$ and conclude that ν goes to the Dirac measure δ_{0^∞}.

The conformal property yields

$$\nu([2]) = \nu([20]) + \nu([21]) + \nu([22]) = e^{-P-\alpha\beta}(\nu[0] + \nu[1] + \nu[2]) = e^{-P-\alpha\beta}. \tag{19}$$

On the other hand the solution of the system obtained in the proof of Proposition 3.4 shows that

$$\begin{aligned}
\nu([0]) &= \frac{1 + e^{-P-3\beta}F(P,3\beta)}{1 - e^{-2P}F(P,\beta)F(P,3\beta)e^{-4\beta}} F(P,\beta)e^{-P-\beta}\nu([2]), \\
\nu([1]) &= \frac{1 + e^{-P-\beta}F(P,\beta)}{1 - e^{-2P}F(P,\beta)F(P,3\beta)e^{-4\beta}} F(P,3\beta)e^{-P-3\beta}\nu([2]).
\end{aligned}$$

Using the formula $\nu([0]) + \nu([1]) + \nu([2]) = 1$ we get another expression for $\nu([2])$:

$$\begin{aligned}
1 &= \nu([2]) \Bigg(1 + \frac{1 + e^{-P-3\beta}F(P,3\beta)}{1 - e^{-2P}F(P,\beta)F(P,3\beta)e^{-4\beta}} F(P,\beta)e^{-P-\beta} + \\
&\qquad \frac{1 + e^{-P-\beta}F(P,\beta)}{1 - e^{-2P}F(P,\beta)F(P,3\beta)e^{-4\beta}} F(P,3\beta)e^{-P-3\beta}\Bigg) \\
&= \nu([2]) \left(\frac{1 + e^{-P-\beta}F(P,\beta) + e^{-P-3\beta}F(P,3\beta) + e^{-2P-4\beta}F(P,\beta)F(P,3\beta)}{1 - e^{-2P}F(P,\beta)F(P,3\beta)e^{-4\beta}}\right).
\end{aligned} \tag{20}$$

Lemma 3.2 and Proposition 2.4 show that whatever the value of α is, $e^{-P-3\beta}F(P,3\beta)$ goes to 0 as β goes to $+\infty$. On the other hand, $e^{-P-\beta}F(P,\beta)$ is exponentially big (of order e^{β} if α is bigger than 1 and $e^{\alpha\beta}$ if α is smaller than 1). Remember that Equation (19) shows that $\nu([2])$ goes exponentially fast to 0 with exponential speed $-\alpha\beta$.

Lemma 3.6. *If $\alpha > 1$, we get $\lim_{\beta\to+\infty} P(\beta)e^{2\beta} = 1$. For $\alpha = 1$, $P(\beta)e^{2\beta}$ goes to $\frac{1+\sqrt{5}}{2}$.*

Proof. We first do the case $\alpha > 1$. As we said above, the numerator in the right hand side of (20) has order e^{β}. On the other hand $\nu([2])$ has order $e^{-\alpha\beta}$. Therefore, the denominator of the right hand side of (20) goes to 0 with exponential speed $e^{(1-\alpha)\beta}$. Then, Lemma 3.2 shows that $P(\beta)e^{-2\beta}$ goes to 1.

Let us now deal with the case $\alpha = 1$. Copying what we did above we get

$$e^P = \frac{e^{-2\beta}}{P} \frac{1 + \varepsilon_1(\beta)}{1 - e^{-2P}\left(\frac{e^{-2\beta}}{P}\right)^2 (1 + \varepsilon_2(\beta))},$$

with $\varepsilon_i(\beta)$ going to 0 as β goes to $+\infty$. Let l be any accumulation point for $Pe^{2\beta}$. Thus the above expression yields

$$1 = \frac{\frac{1}{l}}{1 - \frac{1}{l^2}} = \frac{l}{l^2 - 1}.$$

This yields $l = \frac{1+\sqrt{5}}{2}$. □

Corollary 3.7. *As β goes to $+\infty$, the ratio $\dfrac{\nu[0*_0]}{\nu[1*_1]}$ goes to 1 for $\alpha > 1$, to $\frac{\sqrt{5}+1}{2}$ for $\alpha = 1$ and to $+\infty$ for $\alpha < 1$. The convergence is non-exponential for $\alpha \geq 1$ and has exponential speed $1-\alpha$ if $\alpha < 1$.*

Proof. We remind that Equation (18) gives

$$\frac{\nu[0*_0]}{\nu[1*_1]} = e^{\beta}\,\frac{(\,1+e^{-P-3\beta}\,F(P,3\,\beta)\,)}{(\,1+e^{-P-\beta}\,F(P,\beta)\,)}.$$

We already know that $e^{-3\beta}F(P,3\beta)$ goes to 0 as β goes to $+\infty$. The denominator has for dominating term $\frac{e^{-\beta}}{P}$. For $\alpha < 1$ we directly get that $\dfrac{\nu[0*_0]}{\nu[1*_1]}$ goes to $+\infty$. For $\alpha \geq 1$ we use Lemma 3.6. □

Equation 19 shows that $\nu([2])$ goes to 0 as β goes to $+\infty$. Then Proposition 3.4 yields:

Corollary 3.8. *The measure ν goes to the Dirac measure δ_{0^∞} as β goes to $+\infty$.*

4 The eigenfunction H

In this section we get estimates at the non-exponential scale for the asymptotics behavior of the eigenfunction H_β. In what follows, for simplicity, we will drop the subscriptβ.

4.1 The exponential scale is not deterministic

We know that

$$H(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\frac{\mathcal{L}^k(\mathbb{1})(x)}{e^{kP}} \tag{21}$$

where $\mathcal{L}$ is the transfer operator (see Subsection 1.2). We recall that $*_0$ (resp. $*_1$) denotes any symbol different to 0 (resp. to 1). We start with the following result.

Lemma 4.1. *The eigenfunction is constant on cylinders $[0^n*_0]$, $[1^n*_1]$ and $[2]$.*

Proof. Owing to Equation 21, it is sufficient to prove that for every k, $\mathcal{L}^k(\mathbb{1})$ is constant on cylinders $[0^n*_0]$, $[1^n*_1]$ and $[2]$. For x in Σ, we get

$$\mathcal{L}_\beta^k(\mathbb{1})(x) = \sum_{z\in\{0,1,2\}^k} e^{\beta.S_k(A)(zx)},$$

where $S_k(A)$ is the Birkhoff sum $A + A\circ\sigma + \ldots + A\circ^{k-1}\sigma$. Now, note that the potential is constant on the cylinders $[0^m*_0]$, $[1^m*_1]$ (whatever $m \geq 1$ is) and $[2]$. This finishes the proof of the lemma. □

We emphasize here that the information we get on the subaction (namely the exponential asymptotic for H) and on the eigenmeasure are not yet sufficient to conclude the proof. Indeed, one important fact is that the eigenmeasure and the eigenfunction have opposite behavior: the eigenmeasure is exponentially bigger close to 0^∞ than close to 1^∞; on the contrary, the eigenfunction is exponentially bigger close to1^∞ than close to 0^∞. The convergence and the study of selection for μ cannot be obtained at the exponential scale:

Lemma 4.2. *For $\alpha \geq 1$ and for every integer $n \geq 1$,* $\lim_{\beta \to +\infty} \frac{1}{\beta} \log \frac{\mu([0^n *_0])}{\mu([1^n *_1])} = 0.$

Proof. By definition we get $\frac{\mu([0^n *_0])}{\mu([1^n *_1])} = \frac{H(0^n *_0)\nu([0^n *_0])}{H(1^n *_1)\nu([1^n *_1])}$. Using Corollaries 3.5 and 3.7, we get that $\frac{1}{\beta} \log \frac{\nu([0^n *_0])}{\nu([1^n *_1])}$ goes to $1 - \frac{1}{2^{n-1}}$ as β goes to $+\infty$.

On the other hand, Lemma 2.3 and Proposition 2.4 shows that $\frac{1}{\beta} \log \frac{H(0^n *_0)}{H(1^n *_1)}$ goes to $-1 + \frac{1}{2^{n-1}}$ as β goes to $+\infty$. Both terms balance themselves. □

Remark 2. For $\alpha < 1$ it is also possible to show, following the same procedure, that $\lim_{\beta \to +\infty} \frac{1}{\beta} \log \frac{\mu([0^n *_0])}{\mu([1^n *_1])} = 2 - 2\alpha$.

As the convergence and the study of selection for μ cannot be obtained at the exponential scale we must get more precise estimates.

4.2 Estimation at the non-exponential scale

We recall that the functions $F(P, \beta)$ and $F_n(P, \beta)$ were defined in Definition 3.1.

Lemma 4.3. *For every $n \geq 1$ we get*

$$H(0^n *_0) = e^{(n-1)P - \frac{\beta}{2^n}} \frac{(e^P - 1)}{e^P + e^{-\alpha\beta}} \left[e^{P+\beta} H(1^\infty) - \left(F_{n-2}(P, \beta)(1 + e^{-P-\alpha\beta}) + e^{(1-\alpha)\beta} \right) H(0^\infty) \right], \tag{22}$$

$$H(1^n *_1) = e^{(n-1)P - \frac{3\beta}{2^n}} \frac{(e^P - 1)}{e^P + e^{-\alpha\beta}} \left[e^{P+3\beta} H(0^\infty) - \left(F_{n-2}(P, 3\beta)(1 + e^{-P-\alpha\beta}) + e^{(3-\alpha)\beta} \right) H(1^\infty) \right], \tag{23}$$

where $F_{-1} \equiv 0$.

Proof. Using the equality $\mathcal{L}(H) = e^P H$ we get the following system of equations

$$\begin{cases} e^{-\frac{\beta}{2}} H(0*_0) & & +e^{-\alpha\beta} H(2) & = & (e^P - 1) H(1^\infty), \\ & e^{-\frac{3\beta}{2}} H(1*_1) & +e^{-\alpha\beta} H(2) & = & (e^P - 1) H(0^\infty), \\ e^{-\frac{\beta}{2}} H(0*_0) & +e^{-\frac{3\beta}{2}} H(1*_1) & +(e^{-\alpha\beta} - e^P) H(2) & = & 0. \end{cases} \tag{24}$$

Solving this system in terms of $H(1^\infty)$ and $H(0^\infty)$ we find:

$$H(0*_0) = e^{\frac{\beta}{2}} \frac{(e^P - 1)}{e^P + e^{-\alpha\beta}} \left[e^P H(1^\infty) - e^{-\alpha\beta} H(0^\infty) \right] \tag{25}$$

$$H(1*_1) = e^{\frac{3\beta}{2}} \frac{(e^P - 1)}{e^P + e^{-\alpha\beta}} \left[e^P H(0^\infty) - e^{-\alpha\beta} H(1^\infty) \right] \tag{26}$$

Again, the equality $\mathcal{L}(H) = e^p H$ yields

$$e^P H(0^n *_0) = e^{-\frac{\beta}{2^{n+1}}} H(0^{n+1} *_0) + e^{-\frac{3\beta}{2}} H(1*_1) + e^{-\alpha\beta} H(2).$$

Introducing the second equation in (24), we get

$$H(0^{n+1}*_0) = e^{P+\frac{\beta}{2^{n+1}}} H(0^n*_0) - e^{\frac{\beta}{2^{n+1}}} (e^P - 1) H(0^\infty).$$

By induction, we get for every $n \geq 2$ an expression of $H(0^n*_0)$ in function of $H(0^\infty)$ and $H(0*_0)$. Then, introducing (25) in this expression, we let the reader check that we get (22). The proof of (23) is similar. □

As we said above, the exponential scale is not sufficient to determine the limit and the selection for the Gibbs measure. Due to the values of the subactions, the good parameter to estimate is $e^\beta \dfrac{H(0^\infty)}{H(1^\infty)}$. Lemma 4.3 allows us to solve that problem.

Proposition 4.4. *As β goes to $+\infty$ we get the following limits:*

(i) if $\alpha > 1$, then, $\displaystyle\lim_{\beta\to+\infty} e^\beta \frac{H(0^\infty)}{H(1^\infty)} = 1$,

(ii) if $\alpha = 1$, then, $\displaystyle\lim_{\beta\to+\infty} e^\beta \frac{H(0^\infty)}{H(1^\infty)} = \frac{1+\sqrt{5}}{2}$,

(iii) if $0 < \alpha < 1$, then, $\displaystyle\lim_{\beta\to+\infty} e^\beta \frac{H(0^\infty)}{H(1^\infty)} = +\infty$.

Proof. Equalities (22) and (23) yield for any fixed n

$$e^{\beta - \frac{\beta}{2^{n-1}}} \frac{H(0^n*_0)}{H(1^n*_1)} = \frac{e^P - [\, F_{n-2}(P,\beta)\,(1+e^{-P-\alpha\,\beta})\,e^{-2\beta} \, + e^{-(1+\alpha)\beta}\,]\,(e^\beta \, \frac{H(0^\infty)}{H(1^\infty)})}{e^P\,(e^\beta\,\frac{H(0^\infty)}{H(1^\infty)}) \, - \, [\, F_{n-2}(P,3\beta)\,(1+e^{-P-\alpha\,\beta})\,e^{-2\beta} \, + e^{(1-\alpha)\beta}\,]}. \tag{27}$$

For, β fixed, we set $x = x_\beta = e^\beta\,\frac{H(0^\infty)}{H(1^\infty)}$. Then, taking the limit as n goes to $+\infty$ we get

$$x = \frac{e^P - [\, F(P,\beta)\,(1+e^{-P-\alpha\,\beta})\,e^{-2\beta} \, + e^{-(1+\alpha)\beta}\,]\,x}{e^P\,x \, - \, [\, F(P,3\beta)\,(1+e^{-P-\alpha\,\beta})\,e^{-2\beta} \, + e^{(1-\alpha)\beta}\,]},$$

(the eigenfunction is continuous). Let us set $a = d = e^P$ and

$$b = -[\, F(P,\beta)\,(1+e^{-P-\alpha\,\beta})\,e^{-2\beta} \, + e^{-(1+\alpha)\beta}\,],$$
$$c = -[\, F(P,3\beta)\,(1+e^{-P-\alpha\,\beta})\,e^{-2\beta} \, + e^{(1-\alpha)\beta}\,].$$

We can write the above equation in the form

$$x = \frac{a + b\,x}{d\,x + c}.$$

As x is positive we can solve this equation and we get

$$x = \frac{(b-c) \, + \, \sqrt{(c-b)^2 + 4\,a\,d}}{2\,d}. \tag{28}$$

Note that

$$(b-c) = (\, F(P,3\beta) - F(P,\beta)\,)\; e^{-2\,\beta}\,(1+e^{-P-\alpha\,\beta}) + e^{-\alpha\,\beta}\,(e^\beta - e^{-\beta}).$$

Now, Lemma 3.2 shows that $e^{-\,2\,\beta}\,(\,F(P,3\,\beta) - F(P,\beta)\,) \to 0$ when β goes to $+\infty$. On the other hand we get, for $\alpha > 1$, $e^{-\alpha\beta}(e^{\beta} - e^{-\beta}) \to 0$.

for $\alpha < 1$, $e^{-\alpha\beta}(e^{\beta} - e^{-\beta}) \to +\infty$,

for $\alpha = 1$, $e^{-\alpha\beta}(e^{\beta} - e^{-\beta}) \to 1$,

these three limits hold as β goes to $+\infty$. From this, we get that for the three cases of possible values of α, the corresponding limits for $(b - c)$ are the same:

for $\alpha > 1$, $b - c \to 0$.

for $\alpha < 1$, $b - c \to +\infty$,

for $\alpha = 1$, $b - c \to 1$.

Finally, from this we get that for $\alpha > 1$,

$$\lim_{\beta \to +\infty} e^{\beta} \frac{H(0^\infty)}{H(1^\infty)} = 1,$$

for $\alpha = 1$,

$$\lim_{\beta \to +\infty} e^{\beta} \frac{H(0^\infty)}{H(1^\infty)} = \frac{1 + \sqrt{5}}{2},$$

and for $0 < \alpha < 1$,

$$\lim_{\beta \to +\infty} e^{\beta} \frac{H(0^\infty)}{H(1^\infty)} = +\infty.$$

□

5 End of the proof of the Theorem

Now, we can finish the proof of our Main Theorem. We recall that any accumulation point for μ_β is a A-maximizing measure. Hence, such an accumulation point is a convex combination of the two Dirac measures δ_{0^∞} and δ_{1^∞}. This convex combination can be found if we get an estimate for $\lim_{\beta \to +\infty} \dfrac{\mu([0])}{\mu([1])}$. We get

$$\begin{aligned}
\frac{\mu([0])}{\mu([1])} &= \frac{\sum_{n=1}^{+\infty} \mu([0^n *_0])}{\sum_{n=1}^{+\infty} \mu([1^n *_1])} \\
&= \frac{\sum_{n=1}^{+\infty} H(0^n *_0)\nu([0^n *_0])}{\sum_{n=1}^{+\infty} H(1^n *_1)\nu([1^n *_1])} \\
&= \frac{\sum_{n=1}^{+\infty} H(0^n *_0) e^{-(n-1)\,P - \beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}}{\sum_{n=1}^{+\infty} H(1^n *_1) e^{-(n-1)\,P - 3\beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}} \, \frac{\nu([0 *_0])}{\nu([1 *_1])} \\
&= \frac{\sum_{n=1}^{+\infty} \frac{H(0^n *_0) e^{-(n-1)\,P - \beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}}{H(1^n *_1) e^{-(n-1)\,P - 3\beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}\nu([1 *_1])} H(1^n *_1) e^{-(n-1)\,P - 3\beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}\nu([1 *_1])}{\sum_{n=1}^{+\infty} H(1^n *_1) e^{-(n-1)\,P - 3\beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}} \, \frac{\nu([0 *_0])}{\nu([1 *_1])} \\
&= \frac{\sum_{n=1}^{+\infty} e^{\beta(1-\frac{1}{2^{n-1}})} \frac{H(0^n *_0)}{H(1^n *_1)} \mu([1^n *_1])}{\nu([1 *_1]) \sum_{n=1}^{+\infty} H(1^n *_1) e^{-(n-1)\,P - 3\beta\,(\frac{1}{2^2}+\ldots+\frac{1}{2^n})}} \, \frac{\nu([0 *_0])}{\nu([1 *_1])} \\
&= \frac{\sum_{n=1}^{+\infty} e^{\beta(1-\frac{1}{2^{n-1}})} \frac{H(0^n *_0)}{H(1^n *_1)} \mu([1^n *_1])}{\sum_{n=1}^{+\infty} \mu([1^n *_1])} \, \frac{\nu([0 *_0])}{\nu([1 *_1])}. \qquad (29)
\end{aligned}$$

The proof will follow from the next technical lemma:

Lemma 5.1. *There exists β_0 such that for every $n \geq 3$, for every $\beta \geq \beta_0$ and for every α*

$$\left| e^{\beta(1-\frac{1}{2^{n-1}})} \frac{H(0^n *_0)}{H(1^n *_1)} \times \frac{1}{e^{\beta} \frac{H(0^\infty)}{H(1^\infty)}} - 1 \right| \leq e^{-\frac{\beta}{8}}.$$

Proof. We re-employ notations from the proof of Proposition 4.4. We denote by $R_{n-1}(1)$ the tail

$$R_{n-1}(1) = F(P, \beta) - F_{n-2}(P, \beta) = \sum_{k=n-1}^{\infty} e^{-k\,P\; + \frac{\beta}{2^{k+1}}},$$

$R_{n-1}(3)$ the tail

$$R_{n-1}(3) = F(P, 3\,\beta) - F_{n-2}(P, 3\,\beta) = \sum_{k=n-1}^{\infty} e^{-k\,P\; + \frac{3\beta}{2^{k+1}}}$$

and

$$\Delta_{n-1} = R_{n-1}(1) - R_{n-1}(3) = e^{-(n-1)\,P}\,(e^{\frac{\beta}{2^n}} - e^{\frac{3\,\beta}{2^n}}) + \ldots\,.$$

Then,

$$\begin{aligned}
e^{\beta - \frac{\beta}{2^{n-1}}} \frac{H(0^n *_0)}{H(1^n *_1)} &= \frac{a + bx + x\Delta_{n-1} e^{-2\beta}(1 + e^{-P-\alpha\,\beta}) + xR_{n-1}(3) e^{-2\beta}(1 + e^{-P-\alpha\,\beta})}{c + dx + R_{n-1}(3) e^{-2\beta}(1 + e^{-P-\alpha\,\beta})} \\
&= x + \frac{x\Delta_{n-1} e^{-2\beta}(1 + e^{-P-\alpha\,\beta})}{c + dx + R_{n-1}(3)\, e^{-2\beta}(1 + e^{-P-\alpha\,\beta})}. \qquad (30)
\end{aligned}$$

Remember that by definition we have $x = e^{\beta}\dfrac{H(0^\infty)}{H(1^\infty)}$. Now Equation (23) yields

$$\frac{H(1^n*_1)}{H(1^\infty)}\frac{e^P+e^{-\alpha\beta}}{(e^P-1)}e^{-(n-1)P+\frac{3\beta}{2^n}-2\beta} = dx + c + R_{n-1}(3)e^{-2\beta}(1+e^{-P-\alpha\,\beta}).$$

If n goes to $+\infty$ the right hand side term of this equality goes to $c+dx$. On the other side it is always non-negative. This shows that $c+dx$ is always non-negative. Therefore (30) yields

$$\left| e^{\beta(1-\frac{1}{2^{n-1}})}\frac{H(0^n*_0)}{H(1^n*_1)} \times \frac{1}{e^{\beta}\frac{H(0^\infty)}{H(1^\infty)}} - 1 \right| \leq \frac{|\Delta_{n-1}|}{R_{n-1}(3)}.$$

Now, note that $R_{n-1}(1) = F(P, \frac{\beta}{2^{n-1}})$ and $R_{n-1}(3) = F(P, \frac{3\beta}{2^{n-1}})$. Then, Lemma 3.2 shows that $\dfrac{|\Delta_{n-1}|}{R_{n-1}(3)}$ is of order $P(\beta)\dfrac{\beta}{2^n}e^{\frac{3\beta}{2^{n-1}}}$. Remember that $P \in O(e^{-\beta})$. For $n \geq 3$ and for β sufficiently big, $P(\beta)\dfrac{\beta}{2^n}e^{\frac{3\beta}{2^{n-1}}}$ is less than $e^{-\frac{\beta}{8}}$. □

Now Equation (29) and Lemma 5.1 show that we get for every $\beta \geq \beta_0$

$$e^{\beta}\frac{H(0^\infty)}{H(1^\infty)}(1-e^{-\frac{\beta}{8}})\frac{\nu([0*_0])}{\nu([1*_1])} \leq \frac{\mu([0])}{\mu([1])} \leq e^{\beta}\frac{H(0^\infty)}{H(1^\infty)}(1+e^{-\frac{\beta}{8}})\frac{\nu([0*_0])}{\nu([1*_1])},$$

(for β big the terms $\mu([0^k*_0])$ and $\mu([1^k*_1])$, $k = 1, 2$ are very small since μ_β "goes" to a combination of δ_{0^∞} and δ_{1^∞}). Then Corollary 3.7 and Proposition 4.4 conclude.

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