Stochastics and Dynamics
Vol. 13, No. 1 (2013) 1250009 (13 pages)

DOI: 10.1142/S0219493712500098

THE EFFECTIVE POTENTIAL AND TRANSSHIPMENT IN THERMODYNAMIC FORMALISM AT ZERO TEMPERATURE

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Received 4 August 2010
Revised 30 January 2012
Accepted 19 February 2012
Published 14 August 2012

For a topologically transitive subshift of finite type defined by a symmetric transition matrix, we introduce a temperature-based problem related to the usual thermodynamic formalism. This problem is described by an operator acting on Hölder continuous observables which is actually superlinear with respect to the max-plus algebra. We thus show that, for each fixed absolute temperature, such an operator admits a unique eigenfunction and a unique eigenvalue. We also study the convergence as the temperature goes to zero and we relate the limit objects to an ergodic version of Kantorovich transshipment problem.

Keywords: Thermodynamic formalism; effective potential; transshipment; Gibbs state; additive eigenvalue; maximizing probabilities.

AMS Subject Classification: 37A05, 37A35, 37A60, 37D35, 49K27, 49Q20, 82B05, 90C05, 90C50

1. Introduction

Given $A : \{1, \ldots, r\}^2 \to \mathbb{R}$, $\phi : \{1, \ldots, r\} \to \mathbb{R}$, and $\beta > 0$, it is well known that the unique probability vector $(\mu(1), \ldots, \mu(r)) \in [0, 1]^r$ that maximizes the

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expression

$$\sum_{x_0\in\{1,\ldots,r\}} \mu(x_0)[\beta A(y_0,x_0)+\phi(x_0)] - \sum_{x_0\in\{1,\ldots,r\}} \mu(x_0)\log\mu(x_0) \tag{1}$$

is the Gibbs state associated to A at temperature β^{-1} defined by

$$\mu_{y_0,\beta}(x_0) := \frac{1}{Z_{y_0,\beta}} \exp[\beta A(y_0,x_0)+\phi(x_0)] \quad \forall\, x_0 \in \{1,\ldots,r\},$$

where $Z_{y_0,\beta}$ is a normalization factor.

The analysis of the thermodynamic formalism for a given observable A at zero temperature is, by definition, the study of the limit behavior when $\beta \to \infty$ of the corresponding Gibbs states at temperature β^{-1}. For instance, in the previous elementary example of an equilibrium state for a finite system, it is easy to see that the family $\{\mu_{y_0,\beta}\}_{\beta>0}$ converges to the equidistribution on $\operatorname{argmax}\big(A(y_0,\cdot)\big)$.

The aim of this paper is to propose a strategy to contribute to such an analysis when the observable A depends on countably many coordinates, which corresponds, regarding statistical mechanics nomenclature, to the case of long range interactions. The main point of the approach described here is to consider a generalization of expression (1) as an operator acting on potentials ϕ (see Definition 1). In this way, the natural question about the existence of a more suitable potential will have a counterpart in terms of a functional equation (see Theorem 1).

We introduce thus *effective potentials*. The terminology is borrowed from the work of W. Chou and R. Griffiths [4], where, during the study of ground states of one-dimensional systems, the authors realized that, due to interaction and temperature, there exists a particular potential, called effective potential, which plays an essential role in the problem. In [11], questions related also to the article of Chou and Griffiths were analyzed in the context of Markov chains on the interval.

The effective-potential formalism will allow us to present a family of *effective probabilities*, each one corresponding to a Gibbs state in the sense of Ruelle's thermodynamic setting [14]. We will also consider the limit behavior of this family of probabilities when the temperature goes to zero. In this case, we relate our analysis with an ergodic version of Kantorovich transshipment problem. We recall that, in the classical transport theory [15, 17, 18], there is no assumption involving the invariance of probabilities.

The method proposed here has similarities with entropy penalization techniques, which were considered, for instance, in [9] and [8] (see the main properties on these references) in the setting of Aubry–Mather theory. Nevertheless, in this paper, the entropy to be considered is Kolmogorov–Sinai entropy, which has a prominent dynamical character.

The central questions addressed here find analogues in other physical domains (see, for example, Sec. 2.5 of Salmhofer's book [16]). Finally, we emphasize that the relation of the effective action problem with the ergodic Kantorovich transshipment problem (see Sec. 4), as far as we know, is completely new.

2. Setting and Results

Let $\mathbf{M}: \{1, \ldots, r\} \times \{1, \ldots, r\} \to \{0, 1\}$ be an irreducible transition matrix. One naturally has two subshifts associated to such a matrix. We can introduce the standard subshift of finite type

$$\Sigma_{\mathbf{M}} = \{(x_0, x_1, \ldots) \in \{1, \ldots, r\}^{\mathbb{Z}_+} : \mathbf{M}(x_j, x_{j+1}) = 1\},$$

as well as the dual subshift of finite type

$$\Sigma^*_{\mathbf{M}^T} = \{(\ldots, x_1, x_0) \in \{1, \ldots, r\}^{\mathbb{Z}_-} : \mathbf{M}^T(x_j, x_{j+1}) = 1\}.$$

As topological spaces, both subshifts are always compact metrizable spaces. We suppose henceforth that the matrix $\mathbf{M}$ is symmetric. So we have a canonical homeomorphism $\mathbf{x} = (x_0, x_1, \ldots) \in \Sigma_{\mathbf{M}} \mapsto \mathbf{x}^* = (\ldots, x_1, x_0) \in \Sigma^*_{\mathbf{M}}$.

Given $\Lambda \in (0, 1)$, we equip as usual $\Sigma_{\mathbf{M}}$ with the metric $d(\mathbf{x}, \mathbf{y}) = \Lambda^k$, where $\mathbf{x} = (x_0, x_1, \ldots), \mathbf{y} = (y_0, y_1, \ldots) \in \Sigma_{\mathbf{M}}$ and $k = \min\{j : x_j \neq y_j\}$. Hence, for $\mathbf{x}^*, \mathbf{y}^* \in \Sigma^*_{\mathbf{M}}$, we just set $d^*(\mathbf{x}^*, \mathbf{y}^*) := d(\mathbf{x}, \mathbf{y})$.

Let σ be the left shift map acting on $\Sigma_{\mathbf{M}}$ and let σ^* be the right shift map acting on $\Sigma^*_{\mathbf{M}}$, namely,

$$\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots) \quad \text{and} \quad \sigma^*(\ldots, x_2, x_1, x_0) = (\ldots, x_2, x_1).$$

Clearly, $* \circ \sigma = \sigma^* \circ *$. Furthermore, since $\mathbf{M}$ is irreducible, notice that the dynamics $(\Sigma_{\mathbf{M}}, \sigma)$ is transitive – and consequently the conjugated dynamical system $(\Sigma^*_{\mathbf{M}}, \sigma^*)$ too.

Let $C^0(\Sigma_{\mathbf{M}})$ and $C^0(\Sigma^*_{\mathbf{M}})$ denote the spaces of continuous real-valued functions on respectively $\Sigma_{\mathbf{M}}$ and $\Sigma^*_{\mathbf{M}}$, both equipped with the topology of uniform convergence. Thus, we can obtain from the previous homeomorphism an isometry $* : C^0(\Sigma_{\mathbf{M}}) \to C^0(\Sigma^*_{\mathbf{M}})$ writing $f^*(\mathbf{x}^*) := f(\mathbf{x})$ for every function $f \in C^0(\Sigma_{\mathbf{M}})$. This fact allows us to make the identification $C^0(\Sigma_{\mathbf{M}}) \simeq C^0(\Sigma^*_{\mathbf{M}})$.

The same isometric property is verified for either Hölder or Lipschitz continuous real-valued functions. Since one can simply incorporate the Hölder exponent into the distance, we remark that to work with the Lipschitz class does not lead to loss of generality. Therefore, $\mathcal{H}$ will denote in this paper the Banach space of Lipschitz continuous real-valued functions on either $\Sigma_{\mathbf{M}}$ or $\Sigma^*_{\mathbf{M}}$, equipped with the norm $\|\cdot\|_{\mathcal{H}} := \|\cdot\|_0 + \mathrm{Lip}(\cdot)$, where $\|\cdot\|_0$ denotes the uniform norm and

$$\mathrm{Lip}(\phi) = \sup_{d(\mathbf{x},\mathbf{y})>0} \frac{|\phi(\mathbf{x}) - \phi(\mathbf{y})|}{d(\mathbf{x}, \mathbf{y})} = \sup_{d^*(\mathbf{x}^*,\mathbf{y}^*)>0} \frac{|\phi^*(\mathbf{x}^*) - \phi^*(\mathbf{y}^*)|}{d^*(\mathbf{x}^*, \mathbf{y}^*)} = \mathrm{Lip}(\phi^*).$$

Using the standard subshift $\Sigma_{\mathbf{M}}$ and its dual $\Sigma^*_{\mathbf{M}}$, one may easily introduce its natural invertible extension $(\hat{\Sigma}_{\mathbf{M}}, \hat{\sigma})$:

$$\hat{\Sigma}_{\mathbf{M}} = \{(\mathbf{y}^*, \mathbf{x}) \in \Sigma^*_{\mathbf{M}} \times \Sigma_{\mathbf{M}} : \mathbf{M}(y_0, x_0) = 1\},$$

$$\hat{\sigma}(\ldots, y_1, y_0 \,|\, x_0, x_1, \ldots) = (\ldots, y_0, x_0 \,|\, x_1, x_2, \ldots).$$

Denote by $\mathcal{M}_\sigma$ the weak* compact and convex set of σ-invariant Borel probability measures. For $\mu \in \mathcal{M}_\sigma$, let $h_\mu(\sigma)$ indicate the Kolmogorov–Sinai entropy.

Definition 1. Given a Lipschitz continuous function $A:\hat{\Sigma}_{\mathbf{M}} \to \mathbb{R}$, we consider then the map $\mathcal{G}^+ = \mathcal{G}^+_A : \mathcal{H} \to \mathcal{H}$ defined[a] by

$$\mathcal{G}^+(\phi)(\mathbf{y}^*) = \sup_{\mu \in \mathcal{M}_\sigma} \left[\int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y}^*, \mathbf{x}) + \phi(\mathbf{x}))\, d\mu(\mathbf{x}) + h_\mu(\sigma) \right].$$

It is not difficult to see that $\mathrm{Lip}(\mathcal{G}^+(\phi)) \le \|A\|_0 + \mathrm{Lip}(A)$ for all $\phi \in \mathcal{H}$. Moreover, notice that, for all $\phi, \psi \in \mathcal{H}$ and $\gamma \in \mathbb{R}$, clearly $\mathcal{G}^+(\phi + \gamma) = \mathcal{G}^+(\phi) + \gamma$ and $\mathcal{G}^+(\max(\phi, \psi)) \ge \max(\mathcal{G}^+(\phi), \mathcal{G}^+(\psi))$, which means that the operator $\mathcal{G}^+$ is superlinear with respect to the max-plus algebra. Our main result assures then the existence of eigenfunctions and eigenvalue and can be stated as follows.

Theorem 1. *Suppose $A:\hat{\Sigma}_{\mathbf{M}} \to \mathbb{R}$ is a Lipschitz continuous observable. Then there exist a unique function $\phi^+ \in \mathcal{H}$ (up to an additive constant) and a unique constant $\lambda^+ \in \mathbb{R}$ such that*

$$\mathcal{G}^+(\phi^+) = \phi^+ + \lambda^+.$$

We point out that [8, 9, 11] consider a similar problem but for the so-called entropy penalization method. The proof of this theorem will be presented at the end of the paper. Obviously the function ϕ^+ and the constant λ^+ in the previous statement depend on A.

Definition 2. Given a Lipschitz continuous observable $A : \hat{\Sigma}_{\mathbf{M}} \to \mathbb{R}$, we say that a constant $\lambda^+ \in \mathbb{R}$ is the *effective constant* for A if there exists a function $\phi^+ \in \mathcal{H}$ such that $\mathcal{G}^+(\phi^+) = \phi^+ + \lambda^+$. Any such function ϕ^+ is called a (*forward*) *effective potential* for A.

Notice that the characterization via variational principle of the topological pressure $P_{_{TOP}} : \mathcal{H} \to \mathbb{R}$, namely,

$$P_{_{TOP}}(\Psi) = \max_{\mu \in \mathcal{M}_\sigma} \left[\int_{\Sigma_{\mathbf{M}}} \Psi(\mathbf{x})\, d\mu(\mathbf{x}) + h_\mu(\sigma) \right] \quad \forall\, \Psi \in \mathcal{H},$$

implies that

$$\mathcal{G}^+(\phi)(\mathbf{y}^*) = P_{_{TOP}}(A(\mathbf{y}^*, \cdot) + \phi).$$

In particular, thanks to the Ruelle–Perron–Frobenius theorem, for each $\mathbf{y}^* \in \Sigma^*_{\mathbf{M}}$, there exists a unique probability $\mu_{\mathbf{y}^*} \in \mathcal{M}_\sigma$ (the equilibrium state associated to $A(\mathbf{y}^*, \cdot) + \phi \in \mathcal{H}$) achieving the supremum in the definition of the value $\mathcal{G}^+(\phi)(\mathbf{y}^*)$.

Definition 3. For a Lipschitz continuous observable $A : \hat{\Sigma}_{\mathbf{M}} \to \mathbb{R}$ and a point $\mathbf{y}^* \in \Sigma^*_{\mathbf{M}}$, we say that the unique σ-invariant probability $\mu_{\mathbf{y}^*} = \mu_{\mathbf{y}^*, A}$ on $\Sigma_{\mathbf{M}}$ with

$$\int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y}^*, \mathbf{x}) + \phi^+(\mathbf{x})) d\mu_{\mathbf{y}^*}(\mathbf{x}) + h_{\mu_{\mathbf{y}^*}}(\sigma) = \phi^+(\mathbf{y}^*) + \lambda^+$$

[a]Notice that a more rigorous definition would consider $\int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y}^*, \mathbf{x})\mathbf{M}(\mathbf{y}^*, \mathbf{x}) + \phi(\mathbf{x})) d\mu(\mathbf{x})$, where $\mathbf{M}(\mathbf{y}^*, \mathbf{x}) := \mathbf{M}(y_0, x_0)$ for any point $(\mathbf{y}^*, \mathbf{x}) = (\ldots, y_1, y_0 | x_0, x_1, \ldots)$. We prefer to simplify the notation.

is the *effective probability* for A at $\mathbf{y}^*$, where ϕ^+ and λ^+ are the effective ones associated to A. In this way, we get a family of Gibbs states on the variable $\mathbf{x}$ indexed by $\mathbf{y}^*$.

For a fixed A as above, we consider a positive parameter β, the observable βA, and the corresponding ϕ_β^+, λ_β^+ and $\{\mu_{\mathbf{y}^*,\beta A}\}_{\mathbf{y}^*\in\Sigma_{\mathbf{M}}^*}$. We investigate then the limit problem when $\beta \to \infty$, showing the existence (in the uniform topology) of accumulation Lipschitz functions for the family $\{\phi_\beta^+/\beta\}_{\beta>0}$, characterizing the accumulation probabilities of $\{\mu_{\mathbf{y}^*,\beta A}\}_{\beta>0}$ for each $\mathbf{y}^*$, and proving that λ_β^+/β converges (see Sec. 3).

We remark at last that one could also consider the (backward) transformation $\mathcal{G}^- = \mathcal{G}_A^- : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{G}^-(\phi)(\mathbf{x}) = \sup_{\mu\in\mathcal{M}_{\sigma^*}} \left[\int_{\Sigma_{\mathbf{M}}^*} (A(\mathbf{y}^*,\mathbf{x}) + \phi(\mathbf{y}^*))d\mu(\mathbf{y}^*) + h_\mu(\sigma^*)\right]$$

and all analogous results could be easily stated and similarly proved.

The structure of the paper is the following: in Sec. 3 we discuss the thermodynamic properties of the effective objects, in Sec. 4 we consider the ergodic Kantorovich transshipment problem (which appears in a natural way when the temperature goes to zero), and finally in Sec. 5 we present the proof of the main theorem.

3. Thermodynamic Formalism at Zero Temperature

We will analyze the Gibbs probabilities for βA when $\beta \to \infty$. From now on, $\mathbf{y}^*$ is simply denoted by $\mathbf{y}$ and we identify the spaces $\Sigma_{\mathbf{M}}$ and $\Sigma_{\mathbf{M}}^*$. For each real value β, we consider the map $\mathcal{G}_{\beta A}^+ : \mathcal{H} \to \mathcal{H}$ and the corresponding Lipschitz function ϕ_β^+, the forward effective potential for βA, and the corresponding constant $\lambda_\beta^+ \in \mathbb{R}$. For each $\mathbf{y}$, we consider then the effective probability $\mu_{\mathbf{y},\beta A}$ as before. In order to avoid a heavy notation, we will drop A and $+$ in this section.

In this way, for each parameter β, we have the equation

$$\mathcal{G}_\beta(\phi_\beta) = \phi_\beta + \lambda_\beta.$$

Recall that, for each $\mathbf{y}$, we have $\mathcal{G}_\beta(\phi_\beta)(\mathbf{y}) = P_{TOP}(\beta A(\mathbf{y},\cdot) + \phi_\beta)$, where the pressure is considered for the setting in the variable $\mathbf{x}$. Therefore, for each $\mathbf{y}$ and β, one verifies

$$\phi_\beta(\mathbf{y}) + \lambda_\beta = P_{TOP}(\beta A(\mathbf{y},\cdot) + \phi_\beta).$$

The well-known continuity of the topological pressure, in particular, gives us[b]

$$\begin{aligned} &\|P_{TOP}(\beta A(\mathbf{y},\cdot) + \psi) - P_{TOP}(\beta A(\bar{\mathbf{y}},\cdot) + \bar{\psi})\|_0 \\ &\quad \le \beta\|A(\mathbf{y},\cdot)\mathbf{M}(\mathbf{y},\cdot) - A(\bar{\mathbf{y}},\cdot)\mathbf{M}(\bar{\mathbf{y}},\cdot)\|_0 + \|\psi - \bar{\psi}\|_0. \end{aligned} \tag{2}$$

[b]Recall footnote a.

It is then easy to see that $\mathrm{Lip}(\phi_\beta/\beta) \le \|A\|_0 + \mathrm{Lip}(A)$, from which we obtain the following proposition.

Proposition 2. *The family* $\{\phi_\beta/\beta\}$ *is equi-Lipschitz.*

Remember that the effective potential is unique up to an additive constant. So we will consider the following condition: we fix a point $\mathbf{y}^0 \in \Sigma^*_{\mathbf{M}}$ and assume that $\phi_\beta(\mathbf{y}^0) = 0$ for all β. Via subsequences $\beta_n \to \infty$, with $n \to \infty$, using the previous proposition, we get by the Arzela–Ascoli theorem that there exists a continuous function $V : \Sigma^*_{\mathbf{M}} \to \mathbb{R}$ such that $V(\mathbf{y}^0) = 0$ and, in the uniform convergence,

$$\frac{\phi_{\beta_n}}{\beta_n} \to V.$$

Since $\mathrm{Lip}(\phi_\beta/\beta) \le \|A\|_0 + \mathrm{Lip}(A)$ implies $\mathrm{Lip}(V) \le \|A\|_0 + \mathrm{Lip}(A)$, the function V is actually Lipschitz continuous. Notice that, in principle, such a limit could depend on the chosen subsequence.

Proposition 3. *Suppose that in the uniform convergence* $\phi_{\beta_n}/\beta_n \to V$, *when* $\beta_n \to \infty$. *Let* $\mu_{\mathbf{y},\beta_n}$ *be the effective probability for the observable* $\beta_n A$ *at a fixed point* $\mathbf{y}$. *Then, any accumulation probability measure* $\mu^\infty_{\mathbf{y}} \in \mathcal{M}_\sigma$ *of the sequence* $\mu_{\mathbf{y},\beta_n}$ *is a maximizing probability for* $A(\mathbf{y},\cdot) + V$, *that is,*

$$\int (A(\mathbf{y},\cdot) + V)\, d\mu^\infty_{\mathbf{y}} = \max_{\mu \in \mathcal{M}_\sigma} \int (A(\mathbf{y},\cdot) + V)\, d\mu.$$

Proof. Take any σ-invariant probability μ. Thus

$$\int (\beta_n A(\mathbf{y},\cdot) + \phi_{\beta_n})\, d\mu + h_\mu(\sigma) \le P_{\scriptscriptstyle TOP}(\beta_n A(\mathbf{y},\cdot) + \phi_{\beta_n})$$

$$= \int (\beta_n A(\mathbf{y},\cdot) + \phi_{\beta_n})\, d\mu_{\mathbf{y},\beta_n} + h_{\mu_{\mathbf{y},\beta_n}}(\sigma).$$

Given an accumulation probability measure $\mu^\infty_{\mathbf{y}}$ of the sequence $\mu_{\mathbf{y},\beta_n}$, from

$$\int \left(A(\mathbf{y},\cdot) + \frac{\phi_{\beta_n}}{\beta_n} \right) d\mu + \frac{1}{\beta_n} h_\mu(\sigma) \le \int \left(A(\mathbf{y},\cdot) + \frac{\phi_{\beta_n}}{\beta_n} \right) d\mu_{\mathbf{y},\beta_n} + \frac{1}{\beta_n} h_{\mu_{\mathbf{y},\beta_n}}(\sigma),$$

we get the inequality

$$\int (A(\mathbf{y},\cdot) + V)\, d\mu \le \int (A(\mathbf{y},\cdot) + V)\, d\mu^\infty_{\mathbf{y}}.$$

Therefore, $\mu^\infty_{\mathbf{y}}$ is a maximizing probability for $A(\mathbf{y},\cdot) + V$. □

Proposition 4. *Assume that in the uniform convergence* $\phi_{\beta_n}/\beta_n \to V$ *when* $\beta_n \to \infty$. *Suppose also that* $\mu_{\mathbf{y},\beta_n}$, *the effective probability for the observable* $\beta_n A$ *at a* *fixed point* $\mathbf{y}$*, converges in the weak* topology to* $\mu_{\mathbf{y}}^{\infty} \in \mathcal{M}_\sigma$*. Then,*

$$\lim_{n\to\infty} \frac{\lambda_{\beta_n}}{\beta_n} = \max_{\mu\in\mathcal{M}_\sigma} \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}))d\mu(\mathbf{x})$$

$$= \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}))d\mu_{\mathbf{y}}^{\infty}(\mathbf{x}).$$

Proof. As for any given point $\mathbf{y}$

$$\phi_{\beta_n}(\mathbf{y}) + \lambda_{\beta_n} = \int (\beta_n A(\mathbf{y},\cdot) + \phi_{\beta_n})d\mu_{\mathbf{y},\beta_n} + h_{\mu_{\mathbf{y},\beta_n}}(\sigma),$$

then dividing this expression by β_n, taking limit, and using the last proposition, we immediately get the claim. □

We point out that obviously the limit function $V \in \mathcal{H}$ and the limit measure $\mu_{\mathbf{y}}^{\infty} \in \mathcal{M}_\sigma$ may depend on the particular choice of the sequence β_n. Yet the previous proposition shows that the value $\int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\cdot) + V - V(\mathbf{y}))d\mu_{\mathbf{y}}^{\infty}$ does not depend on the point $\mathbf{y}$. Actually it does not even depend on the function V.

Proposition 5. *Suppose that in the uniform convergence* $\phi_{\beta_n}/\beta_n \to V$ *and* $\phi_{\bar\beta_n}/\bar\beta_n \to \overline{V}$ *when* $\beta_n, \bar\beta_n \to \infty$*. Then, for all point* $\mathbf{y}$*,*

$$\max_{\mu\in\mathcal{M}_\sigma} \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}))d\mu(\mathbf{x})$$

$$= \max_{\mu\in\mathcal{M}_\sigma} \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\mathbf{x}) + \overline{V}(\mathbf{x}) - \overline{V}(\mathbf{y}))d\mu(\mathbf{x}).$$

Proof. Passing to subsequences if necessary, we use the previous proposition to define $c := \lim_{n\to\infty} \lambda_{\beta_n}/\beta_n$ and $\bar c := \lim_{n\to\infty} \lambda_{\bar\beta_n}/\bar\beta_n$. Notice that, again from Proposition 4,

$$V(\mathbf{y}) + c = \max_{\mu\in\mathcal{M}_\sigma} \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}))d\mu(\mathbf{x}) \quad \text{and}$$

$$\overline{V}(\mathbf{y}) + \bar c = \max_{\mu\in\mathcal{M}_\sigma} \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y},\mathbf{x}) + \overline{V}(\mathbf{x}))d\mu(\mathbf{x}),$$

for all point $\mathbf{y}$. Let $\mathbf{y}^0$ be a global maximum point for $V - \overline{V}$. Consider then a probability $\mu_0 \in \mathcal{M}_\sigma$ such that $V(\mathbf{y}^0) + c = \int_{\Sigma_{\mathbf{M}}} (A(\mathbf{y}^0,\mathbf{x}) + V(\mathbf{x}))d\mu_0(\mathbf{x})$. It clearly follows that

$$V(\mathbf{y}^0) + c - \overline{V}(\mathbf{y}^0) - \bar c \leq \int_{\Sigma_{\mathbf{M}}} [(A(\mathbf{y}^0,\mathbf{x}) + V(\mathbf{x})) - (A(\mathbf{y}^0,\mathbf{x}) + \overline{V}(\mathbf{x}))]d\mu_0(\mathbf{x})$$

$$= \int_{\Sigma_{\mathbf{M}}} (V(\mathbf{x}) - \overline{V}(\mathbf{x}))d\mu_0(\mathbf{x}) \leq V(\mathbf{y}^0) - \overline{V}(\mathbf{y}^0),$$

which shows that $c \leq \bar c$. We can proceed in the same way changing in the reasoning V and $\overline{V}$. Therefore $c = \bar c$. □

Theorem 6. *There exists the limit* $c_A := \lim_{\beta\to\infty} \lambda_\beta/\beta$.

Proof. The previous propositions guarantee that $\{\lambda_\beta/\beta\}_{\beta>0}$ has a unique accumulation point as β goes to infinity. □

In the next section, we explain how the real constant c_A is related with an ergodic Kantorovich transshipment problem.

4. Ergodic Transshipment

We remark that one may write, for all limit function $V \in \mathcal{H}$ and for any point $\mathbf{y}$,

$$c_A = \max_{\mu\in\mathcal{M}_\sigma} \int_{\Sigma_\mathbf{M}} (A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}))\, d\mu(\mathbf{x}). \tag{3}$$

Therefore, from ergodic optimization theory, one obtains that

$$c_A = \inf_{f\in\mathcal{H}} \sup_{\mathbf{x}} [A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}) + f(\mathbf{x}) - f(\sigma(\mathbf{x}))].$$

Moreover, if we fixed a limit function $V \in \mathcal{H}$, for each point $\mathbf{y}$, there exists a function $U_\mathbf{y} \in \mathcal{H}$ (called a *sub-action with respect to* $A(\mathbf{y},\cdot) + V - V(\mathbf{y})$) such that

$$A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}) + U_\mathbf{y}(\mathbf{x}) - U_\mathbf{y}(\sigma(\mathbf{x})) \le c_A, \quad \forall\, \mathbf{x} \in \Sigma_\mathbf{M}, \tag{4}$$

and the equality holds on the support of the maximizing measure $\mu_\mathbf{y}^\infty$. We refer the reader to [6, 7, 10] for details on ergodic optimization theory.

Equation (4) implies that $V(\mathbf{y}) + c_A \ge V(\mathbf{x}) + A(\mathbf{y},\mathbf{x}) + U_\mathbf{y}(\mathbf{x}) - U_\mathbf{y}(\sigma(\mathbf{x}))$, for all $(\mathbf{y},\mathbf{x}) \in \hat{\Sigma}_\mathbf{M}$. Furthermore, since the equality holds at $(\mathbf{y},\mathbf{x})$ whenever $\mathbf{x}$ belongs to the support of $\mu_\mathbf{y}^\infty$, one has

$$V(\mathbf{y}) + c_A = \sup_{\mathbf{x}} \left[V(\mathbf{x}) + A(\mathbf{y},\mathbf{x}) + U_\mathbf{y}(\mathbf{x}) - U_\mathbf{y}(\sigma(\mathbf{x}))\right], \quad \forall\, \mathbf{y} \in \Sigma_\mathbf{M}^*. \tag{5}$$

This equation clearly underlines another max-plus eigenvalue problem. See, for instance, [1, 3, 5] for more details on such an issue. We get from the above equation that V is an additive eigenfunction and c_A is an additive eigenvalue for the transformation with kernel map

$$\mathcal{C}(\mathbf{y},\mathbf{x}) := A(\mathbf{y},\mathbf{x}) + U_\mathbf{y}(\mathbf{x}) - U_\mathbf{y}(\sigma(\mathbf{x})), \quad \forall\, (\mathbf{y},\mathbf{x}) \in \hat{\Sigma}_\mathbf{M}.$$

By its very construction, the map $(\mathbf{y},\mathbf{x}) \mapsto U_\mathbf{y}(\mathbf{x})$ may depend on the fixed limit function V. Moreover, we only have information on its Lipschitz regularity on the $\mathbf{x}$ variable. In particular, one cannot say *a priori* how the map $(\mathbf{y},\mathbf{x}) \mapsto \mathcal{C}(\mathbf{y},\mathbf{x})$ varies.

However, it is not difficult to provide examples of observables defining a continuous application $\mathcal{C}$ as above. For instance, considering any $A_1, A_2 \in \mathcal{H}$, this is the case for the observable $A(\mathbf{y},\mathbf{x}) = A_1(\mathbf{x}) + A_2(\mathbf{y}), \forall\, (\mathbf{y},\mathbf{x}) \in \hat{\Sigma}_\mathbf{M}$. Indeed, if $V \in \mathcal{H}$ is any possible limit function, let $U \in \mathcal{H}$ be a sub-action with respect to $A_1 + V$, that is: $A_1(\mathbf{x}) + V(\mathbf{x}) + U(\mathbf{x}) - U(\sigma(\mathbf{x})) \le \max_{\mu\in\mathcal{M}_\sigma} \int (A_1 + V) d\mu, \forall\, \mathbf{x} \in \Sigma_\mathbf{M}$. From (3), we then get $A(\mathbf{y},\mathbf{x}) + V(\mathbf{x}) - V(\mathbf{y}) + U(\mathbf{x}) - U(\sigma(\mathbf{x})) \le c_A$ everywhere on $\hat{\Sigma}_{\mathbf{M}}$. In particular, we may choose $U_{\mathbf{y}} \equiv U$ for all $\mathbf{y}$ in such a situation.

In general, by standard selection arguments (see Sec. 2.1 in [13] and references therein), one may always assure the existence of a family of sub-actions $\{U_{\mathbf{y}}\}_{\mathbf{y}}$ for which the corresponding map $(\mathbf{y},\mathbf{x}) \mapsto \mathcal{C}(\mathbf{y},\mathbf{x})$ is Borel measurable. The main point is to consider just those sub-actions obtained as accumulation functions of eigenfunctions of Ruelle transfer operator when the temperature goes to zero through some fixed sequence (see Proposition 29 in [6]). Note that these eigenfunctions are continuous on the observable. We leave the details to the reader. Finally, it is well known in ergodic optimization theory that these sub-actions have uniformly bounded oscillation. Hence, for each fixed limit function V, there exists a family $\{U_{\mathbf{y}}\}_{\mathbf{y}}$ of sub-actions with respect to $A(\mathbf{y},\cdot) + V - V(\mathbf{y})$ such that the map $(\mathbf{y},\mathbf{x}) \in \hat{\Sigma}_{\mathbf{M}} \mapsto \mathcal{C}(\mathbf{y},\mathbf{x}) = A(\mathbf{y},\mathbf{x}) + U_{\mathbf{y}}(\mathbf{x}) - U_{\mathbf{y}}(\sigma(\mathbf{x}))$ is Borel measurable and bounded.[c]

We consider from now on $\mathcal{C}$ as a bounded measurable cost function in order to introduce a transshipment problem. Let then $\pi : \hat{\Sigma}_{\mathbf{M}} \to \Sigma_{\mathbf{M}}$ and $\pi^* : \hat{\Sigma}_{\mathbf{M}} \to \Sigma^*_{\mathbf{M}}$ be the canonical projections. We are specially interested in the set of Borel probabilities $\hat{\eta}(d\mathbf{y}, d\mathbf{x})$ on $\hat{\Sigma}_{\mathbf{M}}$ verifying $(\pi)_*(\hat{\eta}) = (\pi^*)_*(\hat{\eta})$.

Definition 4. (The Ergodic Kantorovich Transshipment Problem) Given $A : \hat{\Sigma}_{\mathbf{M}} \to \mathbb{R}$ Lipschitz continuous, we define the constant

$$\begin{aligned}\kappa_{\text{erg}} &:= \sup_{(\pi)_*(\hat{\eta}) = (\pi^*)_*(\hat{\eta})} \iint_{\hat{\Sigma}_{\mathbf{M}}} \mathcal{C}(\mathbf{y},\mathbf{x}) d\hat{\eta}(\mathbf{y},\mathbf{x}) \\ &= \sup_{(\pi)_*(\hat{\eta}) = (\pi^*)_*(\hat{\eta})} \iint_{\hat{\Sigma}_{\mathbf{M}}} [A(\mathbf{y},\mathbf{x}) + U_{\mathbf{y}}(\mathbf{x}) - U_{\mathbf{y}}(\sigma(\mathbf{x}))] d\hat{\eta}(\mathbf{y},\mathbf{x}).\end{aligned}$$

An ergodic transshipment measure for A is a probability $\hat{\eta}$ on $\hat{\Sigma}_{\mathbf{M}}$, with $(\pi)_*(\hat{\eta}) = (\pi^*)_*(\hat{\eta})$, that attains such a supremum.

We point out that the classical transport or transshipment problems do not have an intrinsic ergodic nature. Note that $\mathcal{C}$ has a dynamical character. We refer the reader to [15] for general results (not of ergodic nature) on transshipment. In [12] an ergodic transport problem is considered.

We claim that $c_A = \kappa_{\text{erg}}$, or in a more self-contained statement:

Theorem 7. *For the Lipschitz observable $\beta A, \beta > 0$, consider its forward effective potential ϕ^+_β (normalized by $\phi^+_\beta(\mathbf{y}^0) = 0$) and its effective constant λ^+_β. Assume that in the uniform convergence $\phi^+_{\beta_n}/\beta_n \to V$ when $\beta_n \to \infty$. Then there is a family $\{U_{\mathbf{y}}\}_{\mathbf{y}}$ of sub-actions with respect to $A(\mathbf{y},\cdot) + V - V(\mathbf{y})$ such that*

$$\lim_{\beta\to\infty} \frac{\lambda^+_\beta}{\beta} = \sup_{(\pi)_*(\hat{\eta}) = (\pi^*)_*(\hat{\eta})} \iint_{\hat{\Sigma}_{\mathbf{M}}} [\, A(\mathbf{y},\mathbf{x}) - U_{\mathbf{y}}(\mathbf{x}) - U_{\mathbf{y}}(\sigma(\mathbf{x}))] d\hat{\eta}(\mathbf{y},\mathbf{x}).$$

[c]Notice that it obviously follows from (4) that such a map $\mathcal{C}$ is bounded from above.

Proof. We remark that inequality (4) implies that $\kappa_{\rm erg} \le c_A$. Indeed, given any Borel probability $\hat\eta$ on $\hat\Sigma_{\mathbf M}$ such that $(\pi)_*(\hat\eta) = (\pi^*)_*(\hat\eta)$, one clearly has

$$\iint_{\hat\Sigma_{\mathbf M}} [A(\mathbf y,\mathbf x) - U_{\mathbf y}(\mathbf x) - U_{\mathbf y}(\sigma(\mathbf x))]\, d\hat\eta(\mathbf y,\mathbf x)$$

$$= \iint_{\hat\Sigma_{\mathbf M}} [A(\mathbf y,\mathbf x) - U_{\mathbf y}(\mathbf x) - U_{\mathbf y}(\sigma(\mathbf x)) + V(\mathbf x) - V(\mathbf y)] d\hat\eta(\mathbf y,\mathbf x) \le c_A.$$

Recall that functional equation (5) shows the limit V is an additive eigenfunction and the constant c_A is an additive eigenvalue for the transformation with kernel $\mathcal C$. Actually, since $\mathcal C$ is bounded, it is easy to obtain that c_A is uniquely determined by

$$c_A = \sup_{\{\mathbf z^k\}_{k\ge1}} \limsup_{k\to\infty} \frac{\mathcal C(\mathbf z^1,\mathbf z^2) + \mathcal C(\mathbf z^2,\mathbf z^3) + \cdots + \mathcal C(\mathbf z^k,\mathbf z^1)}{k},$$

where the supremum is taken among sequences $\{\mathbf z^k\}$ of points of $\Sigma_{\mathbf M} \simeq \Sigma^*_{\mathbf M}$. See Theorem 2.1 in [1] for a general result. Notice now that, for the Borel probability on $\hat\Sigma_{\mathbf M}$ defined by $\hat\eta_k := \frac1k\delta_{(\mathbf z^1,\mathbf z^2)} + \frac1k\delta_{(\mathbf z^2,\mathbf z^3)} + \cdots + \frac1k\delta_{(\mathbf z^k,\mathbf z^1)}$, one has

$$\frac{\mathcal C(\mathbf z^1,\mathbf z^2) + \mathcal C(\mathbf z^2,\mathbf z^3) + \cdots + \mathcal C(\mathbf z^k,\mathbf z^1)}{k} = \iint_{\hat\Sigma_{\mathbf M}} \mathcal C(\mathbf y,\mathbf x)\, d\hat\eta_k(\mathbf y,\mathbf x).$$

Since $(\pi)_*(\hat\eta_k) = (\pi^*)_*(\hat\eta_k)$ for all $k \ge 1$, it obviously follows that $c_A \le \kappa_{\rm erg}$. □

5. Contraction Properties of $\mathcal G^+$

We would like to discuss now the proof of Theorem 1. We start pointing out an immediate contraction property of $\mathcal G^+$ which also follows from the continuity of the topological pressure (2).

Proposition 8. *For all $\phi,\psi \in \mathcal H$, $\|\mathcal G^+(\phi) - \mathcal G^+(\psi)\|_0 \le \|\phi-\psi\|_0$.*

Let us now identify all functions belonging to $\mathcal H$ which are equal up to an additive constant. So in order to obtain a fine contraction property, we introduce the norm

$$\|\phi\|_c := \inf_{\gamma\in\mathbb R} \|\phi+\gamma\|_0$$

for each equivalence class $\phi \in \mathcal H/\text{constants}$.

Theorem 9. *Consider $\phi,\psi \in \mathcal H$ satisfying $Lip(\phi), Lip(\psi) \le K$ for some fixed constant $K > 0$. Then, there exist constants $C = C(K) > 0$ and $\alpha = \alpha(K) > 0$ such that*

$$\|\mathcal G^+(\phi) - \mathcal G^+(\psi)\|_c \le (1 - C\|\phi-\psi\|_c^\alpha)\|\phi-\psi\|_c.$$

We will need the following lemma.

Lemma 10. *Let $A : \hat{\Sigma}_{\mathbf{M}} \to \mathbb{R}$ be Lipschitz continuous observable. Suppose $\phi \in \mathcal{H}$ satisfies Lip$(\phi) \le K$ for a constant $K > 0$. Given a point $\mathbf{y} \in \Sigma^*_{\mathbf{M}}$, let $\mu_{\mathbf{y}} \in \mathcal{M}_\sigma$ be the equilibrium state associated to $A(\mathbf{y}, \cdot) + \phi \in \mathcal{H}$. Then there exist constants $\Gamma = \Gamma(K) > 0$ and $\alpha = \alpha(K) > 0$ such that, if $B_\rho \subset \Sigma_{\mathbf{M}}$ denotes an arbitrary ball of radius $\rho > 0$,*

$$\mu_{\mathbf{y}}(B_\rho) \ge \Gamma \rho^\alpha.$$

Proof. Let $\mu_\Psi \in \mathcal{M}_\sigma$ be the equilibrium measure associated to $\Psi \in \mathcal{H}$. It is well known that μ_Ψ is a Gibbs state. As a matter of fact, if $\mathbf{x}$ is a point belonging to a ball B_{Λ^n} of radius Λ^n, from the very proof of the Gibbs property one can obtain

$$\exp[-\mathrm{Lip}(\Psi)R(\Lambda) - I_{\mathbf{M}}(\mathrm{Lip}(\Psi) + h_{TOP}(\sigma))S(\Lambda)]$$
$$\le \frac{\mu_\Psi(B_{\Lambda^n})}{\exp\left[\sum_{j=0}^{n-1}(\Psi - P_{TOP}(\Psi)) \circ \sigma^j(\mathbf{x})\right]},$$

where R and S are rational functions with $R(0,1), S(0,1) \subset (0,+\infty)$, $I_{\mathbf{M}}$ is a positive integer depending only on the irreducible transition matrix $\mathbf{M}$ and $h_{TOP}(\sigma)$ denotes the topological entropy. For details we refer the reader to [2, 14].

From the variational principle, one has $\Psi - P_{TOP}(\Psi) \ge -\mathrm{Lip}\Psi - h_{TOP}(\sigma)$. So we get $\exp[-\mathrm{Lip}(\Psi)R(\Lambda) - (\mathrm{Lip}(\Psi) + h_{TOP}(\sigma))(I_{\mathbf{M}}S(\Lambda) + n)] \le \mu_\Psi(B_{\Lambda^n})$. Therefore, applying this inequality to $\Psi = A(\mathbf{y}, \cdot) + \phi$, it is straightforward that

$$\Gamma(K)\Lambda^{n\alpha(K)} \le \mu_{\mathbf{y}}(B_{\Lambda^n}),$$

where

$$\alpha(K) := \frac{\mathrm{Lip}(A) + K + h_{TOP}(\sigma)}{\log \Lambda^{-1}} \quad \text{and}$$
$$\Gamma(K) := \exp[-(\mathrm{Lip}(A) + K)R(\Lambda) - I_{\mathbf{M}}(\mathrm{Lip}(A) + K + h_{TOP}(\sigma))S(\Lambda)]. \qquad \square$$

Proof of Theorem 9. Obviously, for $\phi \in \mathcal{H}$ and $\gamma \in \mathbb{R}$, we have $\|\phi + \gamma\|_c = \|\phi\|_c$. Moreover, given $\phi, \psi \in \mathcal{H}$, there exists $\overline{\gamma} \in \mathbb{R}$ such that $\|\phi - \psi\|_c = \|\phi - \psi + \overline{\gamma}\|_0$.

As $\mathcal{G}$ commutes with constants, replacing ψ by $\psi - \min \psi$ and ϕ by $\phi + \overline{\gamma} - \min \psi$, without loss of generality, we may assume $\min \psi = 0$ and $\|\phi - \psi\|_c = \|\phi - \psi\|_0$. We suppose yet $\phi \neq \psi$, since otherwise there is nothing to argue.

Take then $\mathbf{y} \in \Sigma^*_{\mathbf{M}}$ satisfying $\|\mathcal{G}^+(\phi) - \mathcal{G}^+(\psi)\|_0 = |\mathcal{G}^+(\phi)(\mathbf{y}) - \mathcal{G}^+(\psi)(\mathbf{y})|$. By interchanging the roles of ϕ and ψ if necessary, we suppose that

$$|\mathcal{G}^+(\phi)(\mathbf{y}) - \mathcal{G}^+(\psi)(\mathbf{y})| = \mathcal{G}^+(\phi)(\mathbf{y}) - \mathcal{G}^+(\psi)(\mathbf{y}).$$

Since $\min \psi = 0$, taking any point $\mathbf{x} \in \Sigma_{\mathbf{M}}$, we get

$$\|\phi - \psi\|_c \le \|\phi - \phi(\mathbf{x}) - \psi\|_0 \le \|\phi - \phi(\mathbf{x})\|_0 + \|\psi\|_0 \le \mathrm{Lip}(\phi) + \mathrm{Lip}(\psi) \le 2K.$$

Note that $\|\phi-\psi\|_c = \|\phi-\psi\|_0$ implies $\min(\phi-\psi) = -\max(\phi-\psi)$. In particular, $\min(\phi-\psi) = -\|\phi-\psi\|_c$. So there exists a point $\overline{\mathbf{x}} \in \Sigma_{\mathbf{M}}$ such that

$$(\phi-\psi)(\overline{\mathbf{x}}) = -\|\phi-\psi\|_c < 0.$$

Hence, when $\mathbf{x} \in \Sigma_{\mathbf{M}}$ verifies $d(\mathbf{x},\overline{\mathbf{x}}) \leq \frac{\|\phi-\psi\|_c}{4K}$, we obtain

$$\begin{aligned}\phi(\mathbf{x}) - \psi(\mathbf{x}) &\leq |\phi(\mathbf{x}) - \phi(\overline{\mathbf{x}})| + |\psi(\overline{\mathbf{x}}) - \psi(\mathbf{x})| + (\phi-\psi)(\overline{\mathbf{x}}) \\ &\leq 2K\frac{\|\phi-\psi\|_c}{4K} - \|\phi-\psi\|_c = -\frac{\|\phi-\psi\|_c}{2} < 0. \qquad (6)\end{aligned}$$

Let $\mu_{\mathbf{y}} \in \mathcal{M}_\sigma$ be such that $\mathcal{G}^+(\phi)(\mathbf{y}) = \int_{\Sigma_{\mathbf{M}}}(A(\mathbf{y},\mathbf{x}) + \phi(\mathbf{x}))d\mu_{\mathbf{y}}(\mathbf{x}) + h_{\mu_{\mathbf{y}}}(\sigma)$. Notice that $\mathcal{G}^+(\phi)(\mathbf{y}) - \mathcal{G}^+(\psi)(\mathbf{y}) \leq \int_{\Sigma_{\mathbf{M}}}(\phi(\mathbf{x}) - \psi(\mathbf{x}))d\mu_{\mathbf{y}}(\mathbf{x})$.

Thus, if $B_{\frac{\|\phi-\psi\|_c}{4K}}(\overline{\mathbf{x}})$ denotes the closed ball of radius $\frac{\|\phi-\psi\|_c}{4K} \in (0,1)$ and center $\overline{\mathbf{x}} \in \Sigma_{\mathbf{M}}$, from (6) and Lemma 10, we verify

$$\begin{aligned}\mathcal{G}^+(\phi)(\mathbf{y}) - \mathcal{G}^+(\psi)(\mathbf{y}) &\leq \int_{\Sigma_{\mathbf{M}} - B_{\frac{\|\phi-\psi\|_c}{4K}}(\overline{\mathbf{x}})} (\phi(\mathbf{x}) - \psi(\mathbf{x}))d\mu_{\mathbf{y}}(\mathbf{x}) \\ &\leq \|\phi-\psi\|_0\big(1 - \mu_{\mathbf{y}}\big(B_{\frac{\|\phi-\psi\|_c}{4K}}(\overline{\mathbf{x}})\big)\big) \\ &\leq \|\phi-\psi\|_0(1 - C\|\phi-\psi\|_c^\alpha),\end{aligned}$$

where $C := \Gamma/(4K)^\alpha > 0$. As $\|\mathcal{G}^+(\phi) - \mathcal{G}^+(\psi)\|_c \leq \|\mathcal{G}^+(\phi) - \mathcal{G}^+(\psi)\|_0 = \mathcal{G}^+(\phi)(\mathbf{y}) - \mathcal{G}^+(\psi)(\mathbf{y})$, the proof is complete. □

Then Theorem 1 results directly from Theorem 9, the fact that $\mathrm{Lip}(\mathcal{G}^+(\phi)) \leq \|A\|_0 + \mathrm{Lip}(A)$ for all $\phi \in \mathcal{H}$, and the following fixed point theorem due to D. A. Gomes and E. Valdinoci (for a proof, see Appendix A of [9]).

A Banach–Caccioppoli-type Theorem. *Let* $\mathbf{F}$ *be a closed subset of a Banach space, endowed with a norm* $\|\cdot\|$. *Suppose that* $G : \mathbf{F} \to \mathbf{F}$ *is so that*

$$\|G(\phi) - G(\psi)\| \leq (1 - C\|\phi-\psi\|^\alpha)\|\phi-\psi\|,$$

for all $\phi, \psi \in \mathbf{F}$ *and some given constants* $C, \alpha > 0$. *Then there exists a unique* $\phi^+ \in \mathbf{F}$ *such that* $G(\phi^+) = \phi^+$. *Moreover, given any* $\phi_0 \in \mathbf{F}$, *we have*

$$\phi^+ = \lim_{n\to+\infty} G^n(\phi_0).$$

Acknowledgment

E. G. was supported by PROCAD UNICAMP-UFRGS 162-2007. A. O. L. was partially supported by CNPq, PRONEX — Sistemas Dinâmicos, INCT em Matemática, and beneficiary of CAPES financial support (PROCAD UFRGS-IMPA and PROCAD UNICAMP-UFRGS).

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