Mathl Comput. Modelling Vol. 13, No. 9, pp. 17–32, 1990
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0895-7177/90 $3.00 + 0.00
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DYNAMICS OF REAL POLYNOMIALS ON THE PLANE AND TRIPLE POINT PHASE TRANSITION

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(Received October 1989, in revised form August 1990)

Abstract—We will consider the Generalized Chebyshev polynomials on the plane and show the existence of three equilibrium probabilities for the pressure associated with a distinguished value of an external parameter. The phenomena shows an intriguing analogy with the Potts Model of Statistical Mechanics. The main feature of this model is to explain the sudden magnetization of a Ferromagnetic system at a certain value of the temperature. In the Potts model three equilibrium measures coexist at the transition value. In our case two of these equilibrium probabilities have magnetic (or anti-ferromagnetic) nature, and the other one is not of magnetic nature. This phenomena can be classified as a triple point phase transition.

For temperatures below this distinguished value, there exist just two equilibrium measures, namely the two magnetic and anti-ferromagnetic ones mentioned above

Our main theorem is presented in §2, after we explain some properties of the dynamics of the Generalized Chebyshev Polynomials.

The mathematical model presented here has a strong analogy with the phenomena of triple point transition in a semi-infinite one-dimensional spin-lattice N; with four spin-components in each sitte of the lattice. One should also suppose the existence of an anisotropy in the lattice in such way some of the spin-components (or directions) are not favorable.

Some results related to Yang-Lee zeros are presented in the end of the paper.

We refer the reader to the conclusion at the end of the paper for more comments about the above considerations.

INTRODUCTION

In this note, we will show the existence of the phenomena of triple point phase transition in the thermodynamic formalism of a special kind of real polynomials on $\mathbf{R}^2$.

In a previous work [1], we show the existence of a discontinuous jump from an equilibrium measure to another with a change of an external parameter t. In the distinguished value t_0, two equilibrium measures coexist. This result can be seen as a mathematical model for a sudden magnetization of a semi-infinite unidimensional lattice N [1].

In [1], we considered real Chebyshev polynomials and Lattes rational map to create our model, and also a duality of the setting of pressure and generalized dimension was introduced. (See [1] for details.)

In order to present rigorous results for the generalized dimension, we used results from the setting of pressure. The setting of generalized dimension is closer to the procedure used by physicists because it deals with "observables." It is reasonable to say that the pressure is not an "observable."

The Ising model was created by physicists to explain the sudden magnetization of a lattice with certain arrangement of spins. Two equilibrium measures can coexist in the transition temperature.

This work was partially supported by Air Force Office of Scientific Research.

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It is in general accepted that this phenomena does not happen in the lattice Z. It was shown by M.E. Fisher the existence of the phenomena in the lattice N [2], [3]. This model assumes a wall effect, and this is the reason to consider N instead of Z in [2] and [3].

The shift map in the lattice has dynamical properties that are useful in understanding problems in spin-lattices (see [4]).

In the last decades, several mathematical physicists began to analyze the dynamics of maps as a model for understanding physical problems in lattices.

These maps are considered defined on the line, on the plane, or more generally, on manifolds.

Using analogy with the shift, diffeomorphism should model problems in the lattice Z and n to one maps should model problems in the lattice N with n possible spins-components.

An interesting phenomena that also happens in nature is the triple point transition, that is, a sudden appearance of three different equilibrium probabilities for a certain value of the external parameter.

For understanding this phenomena, physicists introduced the Potts model [5]. It is in general considered for the lattice Z.

A large class of different kinds of triple point phase transition are designated under the title Potts Model. Here we will consider a very specific kind of Potts Model.

Now, let's return to mathematical results. M. E. Hoffman and W. D. Withers analyzed dynamic properties of a two-dimensional generalization of the Chebyshev polynomials [6] and [7]. We refer the reader to these two beautiful papers for several results we will use in our paper. Some of these properties are summarized in §1.

Proceeding in an analogous way as in [1], we will use the above mentioned real polynomials in the plane to show the existence of a transition value of an external parameter where three different equilibrium measures coexist (see §2 for more details). The phenomena analogous to a sudden magnetization and a triple point will appear in the model (see [8] for considerations of physical nature).

We are not sure if the mathematical model presented here can explain concrete physical phenomena. We presented the above considerations of physical nature to explain our motivation for analyzing the mathematical problem we will consider in this paper. We believe, anyway, that the techniques used here can be eventually of some help for understanding questions of theoretical nature of the phenomena of triple point transition.

The dynamics of the polynomials we will consider here (in §1, §2, §3) is concentrated in a Deltoid Region (see Fig. 1). It is intriguing that the paper [8], related with the Potts model, and mentioned in [7], presents also a Deltoid picture, but we cannot see a relation with the results presented here. The Deltoid picture in [8] is in the parameter space and the Deltoid region considered here is in phase space. The model considered in [8] is in the lattice Z, and here the analogy is with the lattice N.

The structure of the paper is the following, in §1 we will mention general properties of the dynamics of the Generalized Chebyshev polynomials. In §2, we will consider the setting of pressure, and, in §3, we will consider the setting of generalized dimension. In §5, we will present a conjecture related to Yang-Lee zeros.

I would like to thank M. Barbosa for some explanations about the Potts Model. I would also like to thank W. D. Withers who explained to me the main features of the Generalized Chebyshev polynomials.

1. ON THE DYNAMICS OF THE GENERALIZED CHEBYSHEV POLYNOMIALS ON THE PLANE

The real polynomials f we will consider here (in §1, §2, §3) are defined on $\mathbf{R}^2$ and they have the dynamics concentrated in the interior of the Deltoid Region Δ shown in Figure 1. The points P_1, P_2 and P_3 are, respectively, 3, $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$ and $-\frac{3}{2}-\frac{3\sqrt{3}}{2}i$.

The results presented here (in §1, §2, §3) apply for the infinite family of polynomials presented in [7]. For each natural number n, there exist one of such polynomials. In this case, the analogy would be with the lattice $\mathbf{N}$ with n^2 spin-components.

This family of polynomials is orthogonal with respect to the measure μ that will be mentioned later here. For the sake of simplicity, we will consider here just the polynomials $f(z) = z^2 - 2\bar{z}$, where we are considering $z \in \mathbf{R}^2$ as a complex number and making product of complex numbers. As usual $\bar{z}$ denotes the conjugated of z.

In terms of real coordinates u, v in $\mathbf{R}$, we have $f(u,v) = (u^2 - v^2 - 2u, 2uv + 2v)$.

This real polynomial f is real algebraic, seen as a function of the real variables u and v. It follows from the Bezout Theorem that the topological entropy of f is smaller or equal to $\log 4$. Therefore, any invariant measure has entropy smaller or equal to $\log 4$ ([4, 9]).

Points outside the Deltoid Region go to ∞ under iterations of f.

The Deltoid region is invariant forward and backward by f.

The equation for the Deltoid curve $\delta\Delta$, the boundary of the Deltoid region is

$$-(z\bar{z})^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27 = 0 \tag{1}$$

There exists a subset of Δ with 2-dimensional Lebesgue measure equal to the area of the Deltoid Region Δ, and such the orbit of each point in this set is dense in the interior of the Deltoid Region.

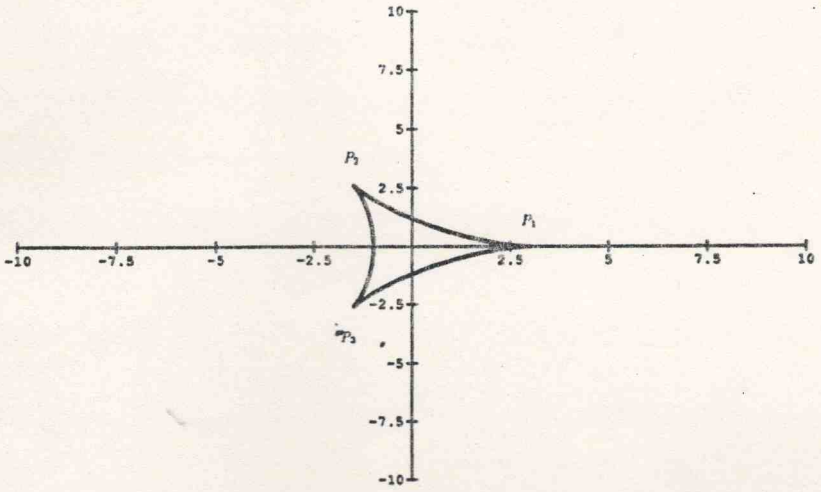

Figure 1.

The Deltoid curve $\delta\Delta$ is invariant forward by f.

The inverse image by f of $\delta\Delta$ is $\delta\Delta$ union with the unit circle. The unit circle is the critical set for f, that is, the set of points in the plane where the Jacobian determinant of f is zero.

The unit circle intercepts the curve $\delta\Delta$ on the points $-1, e^{i\pi/3}$ and $e^{-i\pi/3}$.

The map f restricted to $\delta\Delta$ is a two to one map (counted with multiplicity).

If we denote $B = \Delta - \bigcup_{n=0}^{\infty} f^{-n}(S_1)$, where S_1 denotes the unit circle (the critical set of f), then f restricted to B is four to one.

REMARK 1. The map f is conjugated to the shift map on $\{1,2,3,4\}^{\mathbf{N}}$.

The points P_1, P_2 and P_3, correspond in the spin-lattice $\{1,2,3,4\}^{\mathbf{N}}$, respectively, to the arrangements $(1,1,1,1,\ldots)$, $(2,3,2,3,\ldots)$ and $(3,2,3,2,\ldots)$.

This can be easily seen by means of a change of coordinates h to an equilateral triangle described in [7, page 402].

In these new coordinates, the action of f is described by the map denoted by F_2 in [7]. This map F_2 is a map from the triangle in itself with constant Jacobian determinant equal to 4.

The action of F_2 in the equilateral triangle in the x, y plane is the following: it stretches the triangle by a factor of 2 in the x direction and by a factor of 2 in the y direction and then folds the triangle in itself (see [7]).

There is a natural partition of equilateral triangle in four smaller triangles in such way F_2 is a diffeomorphism of each small triangle in the equilateral triangle. If one gives the labels $\{1,2,3,4\}$ to each one of the four small triangles, then one has a Markov Partition for F_2, that determines the conjugacy with the shift map in $\{1,2,3,4\}^{\mathbf{N}}$ mentioned above (see [7]).

The map h is a diffeomorphism of the interior of the triangle in the Deltoid region and satisfied

$$f = h \circ F_2 \circ h^{-1}.$$

From this last equation it follows that f has constant sum of the Liapunov numbers equal to $\log 4$.

We point out the important point that h has the Derivative Jacobian Matrix in the boundary of the triangle not invertible. The corners of the triangles are associated by h with P_1, P_2 and P_3. In the boundary points the Jacobian determinant of f is not $\log 4$.

The maximal entropy measure μ (see [4, 9] for definitions and general results on Ergodic Theory) has support in the interior of the Deltoid Δ and has a density with respect to the 2-dimensional Lebesgue measure given by

$$\frac{3}{\pi^2}[-(z\bar{z})^2 + 4(z^3 + \bar{z}^3) - 18z\bar{z} + 27]^{-1/2} \tag{2}$$

Note that the density goes to ∞ on the boundary of Δ. The maximal measure μ is ergodic and has entropy $\log 4$. All the above results are presented in [6] and [7].

The measure μ corresponds, in the coordinates $\{1,2,3,4\}^{\mathbf{N}}$, to a Bernoulli system of tossing a coin with 4 faces and with equal probability in each face.

The reason for calling the family of maps f defined on $\mathbf{R}^2$, considered in [6, 7] as Generalized Chebyshev polynomials, is related with orthogonality with respect to the measure μ mentioned above. The real Chebyshev polynomials have this property with respect to a measure on the line.

Now, let's make some more considerations about the dynamics of f. The point P_1 is fixed by f and $\{P_2, P_3\}$ are in an orbit of period 2.

Now let's consider Liapunov numbers of points in the closed region Δ. As we are considering a map on $\mathbf{R}^2$, we have two Liapunov numbers. We will be interested here in the Jacobian determinant, or more precisely, in the sum of the Liapunov numbers. Any point in the set B defined above has sum of the Liapunov numbers equal to $\log 4$. The reason for that is the following: the map F_2 on the equilateral triangles defined on [7] has constant Jacobian determinant 4, and the derivative matrix of the change of coordinates h^{-1} is invertible outside the boundary. This reasoning is the same as used in [1] for the real Chebyshev polynomial $2x^2 - 1$.

Points in the unit circle and in its preimages have Liapunov numbers $-\infty$. There is no invariant measures with support on the set $\Delta - B$. This set is the set of inverse images of the critical set S_1 and has Lebesgue measure zero in the plane.

The sum of Liapunov numbers of points in the Deltoid curve $\delta\Delta$ is also $\log 4$, but the point P_1, P_2 and P_3 has Liapunov numbers equal to $\log 4$ and $\log 8$. The Jacobian determinant of f in the point P_1 is 32 and the Jacobian determinant of f^2 in P_2 and P_3 is 32^2.

REMARK 2. Note that P_1, P_2 and P_3 has the sum of the Liapunov numbers equal to $\log 32$, and therefore strictly larger than $\log 4$, the other possible value for the sum of the Liapunov numbers. This fact will be essential to obtain the results presented here in §2 and §3. Using the terminology of [1], we can say that the map f has a gap (see Def. 22 and Prop. 4. in [1]).

Now, we will explain some other analogies of the real Chebyshev polynomial $2x^2 - 1$ and the map

$$f(z) = z^2 - 2\bar{z}. \tag{3}$$

First note that by means of a linear change of coordinates the Chebyshev polynomial can be written in the form $g(x) = x^2 - 2x$. Note that f restricted to the real line is equal to g. In fact, by means of the change of coordinates h mentioned above, one can obtain the action of g as the restriction of F_2 to a line. This restriction of F_2 is the tent-map with constant modulus of the derivative equal 2. Therefore, one can see f as a two-dimensional version of the embedded action of g in the one-dimensional line.

Now observe that the critical point of the map g is 1, and the fixed points are 0 and 3. A simple calculation shows that $g(1) = -1, g(-1) = 3$ and $g(3) = 3$. The dynamics of g is concentrated in the set [-1,3]. This interval plays for g, the role of the Deltoid region for f. Note that in both cases the critical set has forward image by f and its iterates in the boundary of the region Δ, where the dynamics is concentrated. The maximal entropy measure for g has density

$$\frac{1}{\pi}\frac{1}{\sqrt{(x-3)(x+1)}} \tag{4}$$

Note the singularities of the kind $r^{-1/2}$ in the boundary of the interval [-1,3]. This phenomena also happens for f, but now the boundary is a one-dimensional curve. Using analogy with rational maps one can say that the critical set of f is eventually periodic (see [1]).

The Liapunov number for g of the fixed point 3 is log 4, and the Liapunov number for g of any other point is log 2 or $-\infty$ (see [1]). Therefore, 3 has Liapunov number strictly larger than any other possible one. This situation for g is analogous to what happens for f as we see in Remark 2. The essential difference now is that one has two periodic orbits for f with maximal Liapunov numbers.

In [1], we showed a discontinuous jump from the maximal entropy measure of g to a Delta-Dirac in the point 3. This can be seen as a magnetization and it is related to the Ising Model.

In the situation covered now in this paper, we have a richer situation where the maximal measure for f will coexist as equilibrium measure (for a certain external energy) with the measures δ_{P_1} and $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$, where δ_z denotes the Delta Dirac with mass one in point z.

Remember that by means of a natural change of coordinates (see Remark 1) with the shift in the lattice N (or with the triangle) with 4 possible spin-components {1,2,3,4} (see also [6, 7]), one can see that the point P_1 corresponds to the arrangement (1,1,1,1,...), the point P_2 corresponds to (2,3,2,3,...), and finally P_3 corresponds to (3,2,3,2,...).

One should think a magnetization in the spin-component 1 as a Delta-Dirac in the point (1,1,1,1,...) of the Bernoulli space $\{1,2,3,4\}^{\mathbf{N}}$.

A Delta-Dirac on (2,3,2,3,...) (or on (3,2,3,2,...)) can be seen as an anti-ferromagnetic arrangement.

REMARK 3. In this way we say we have a phenomena of spontaneous magnetization. The maximal measure μ is the measure not magnetic, and δ_{P_1} and $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$ are respectively the other two measures of magnetic and anti-ferromagnetic nature.

The spin-component associated with 4 has no magnetization, because there is no Delta Dirac in the arrangement (4,4,4,4,...). This fact means an anisotropy in this spin-component.

In the next paragraph we will explain the setting of pressure, introducing an external parameter t, such that at the value $t_0 = -\frac{4}{3}$, the system will present the equilibrium probabilities mentioned in Remark 3.

2. THE SETTING OF PRESSURE

We refer the reader to [4, 9–12] for Thermodynamic Formalism Theory of Dynamical Systems. This theory was devised by Bowen, Ruelle and Sinai as a mathematical theory of dynamical system related to ergodic theory and the existence of Gibbs states. We refer the reader to [13, 14]

for a relation of this theory with the Feynman-Kac formula and the Brownian Motion. This last result [13] is also connected with the Large Deviation Theory and the Strook-Varadhan variational formula [15].

In [16], we consider hyperbolic rational maps and showed that, when one considers the maximal entropy measure (see [9, 17] for references), the dimension spectra do not present the phenomena of phase transition. We showed in [1] that phase transitions associated with the generalized dimension do occur for some non-hyperbolic rational maps. One of the examples where such phenomena do occur is the Chebyshev polynomial $2x^2 - 1$ on the real line.

Here we will consider the maximal measure μ for the real polynomial in the plane $f(z) = z^2 - 2\bar{z}$ and show that the phase transition that occurs, in fact, represents the phenomena of triple-point transition. This will be shown in §2 and also in §3, where we will consider the generalized dimension for the maximal measure μ.

There exists a relation between the pressure and the generalized dimension (see [1]). Therefore, we will consider first the setting of pressure, and then in §3, will show that, indeed, the results of pressure are confirmed by the results related with the generalized dimension.

Following Bowen and Ruelle, we will consider for each $t \in \mathbb{R}$ the variational problem

$$\sup_{v \in M(f)} \left\{ h(v) - \frac{t}{2} \int \log |\det(Df(z))| dv(z) \right\},$$

where $M(f)$ denotes the set of invariant probabilities and $h(v)$ is the entropy of $v \in M(f)$ (see [9, 1] for definitions).

We refer the reader to [4, 11] for the relation of this variational problem with Gibbs states.

We will also denote

$$P(t) = \sup_{v \in M(f)} \left\{ h(v) - \frac{t}{2} \int \log |\det(Df(z))| dv(z) \right\}. \tag{5}$$

Note that for $t = 0$, the solution of the variational problem is the maximal measure μ. In the situation covered in this paper, $P(0) = \log 4$, the entropy of f [11].

DEFINITION 1. We will denote, as usual, $P(t)$ as the pressure at the value t. We will denote for each t, the measure $\mu(t)$ as the solution of the variational problem. We will call $\mu(t)$ the equilibrium measure for the external parameter t.

When the map is hyperbolic, it was shown by Bowen [18] (see also [19]) that the parameter value t where $P(t) = 0$ is the value of the Hausdorff dimension of the Julia set.

The function $P(t)$ in this case is convex and real analytic (see [19]). In the hyperbolic case, there is always a unique equilibrium measure for each value t. In the other case the map is not hyperbolic, in some cases, this function $P(t)$ is not differentiable in some points [1]. This phenomena is related to phase transition, and the existence of multiple equilibrium measures [1].

We will show in §2 that for $-\frac{4}{3} < t$, $\mu(t) = \mu$, for $t = -\frac{4}{3}$ we have μ, $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$ and δ_{P_1}, as equilibrium measures, and finally, for $t < -\frac{4}{3}$, we have two equilibrium measures $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$ and δ_{P_1}.

As we mentioned in Remark 3 in §1, this phenomena can be seen as a triple point transition with spontaneous magnetization.

In the next paragraph, we will consider the problem of the point of view of the observables related to the generalized dimension of the maximal measure μ.

We would like to make a comment about a major difference of the procedure followed here (in the approach of Bowen-Ruelle-Sinai [4, 11]) and the one used more often in the analysis of statistical mechanics by physicists (as, for instance, in [8]). In general, physicists consider a partition of the lattice until level N, find the solution at this level, and finally obtain the solution for the infinite lattice considering the thermodynamic limit. We point out that when one considers the more sophisticated concept of entropy of a measure (as we are using here), then one is already

using some kind of information in the limit when N goes to ∞. That is the reason for not existing limits on N in our context.

The point of view used by physicist mentioned above is based on the "observables" of the finite partitions. The concepts in §3 will be related to this point of view. The value t plays here the role of an external temperature (see [11, 13] for more detailed considerations).

MAIN THEOREM. The real polynomial in the plane $f(z) = z^2 - 2\bar{z}$, presents the phenomena of triple point phase transition for the pressure at the parameter $t = -\frac{4}{3}$.

PROOF. We will sketch the proof, because it is very similar to the results shown in [1]. Remember that the sum of the Liapunov numbers in each point P_1, P_2 and P_3 is $\log 32$.

As we mentioned before (see Remark 2), any point in the interior of the Deltoid region Δ has sum of the Liapunov numbers equal to $\log 4$ or $-\infty$.

Invariant measures with support in the boundary of the Deltoid region cannot be analyzed with the previous reasoning because the change of the coordinates h have determinant Jacobian zero in the boundary of the triangle. Therefore we cannot say that the Liapunov number of such measures are smaller than 4.

We will show later that $-\frac{4}{3}$ is the transition value, and therefore, we have to make an specific analysis for the Pressure at this value.

We will assume now that the measures in the boundary will not create any problem in our analysis and follow the argument to show the existence of triple point transition. In the end of this section we will present the argument for the boundary case as claimed above.

Using the terminology of [1], we say that f has a gap. Therefore as μ has maximal entropy, for large values of t

$$h(\mu) - \frac{t}{2}\int \log|\det(Df(z))|\, d\mu(z) = \log 4 - t\log 2 \tag{6}$$

is larger than

$$h(v) - \frac{t}{2}\int \log|\det(Df(z))|\, dv(z)$$

for any other $v \in M(f)$.

In the other way (see [1], Prop. 4) when t is very negative, then, the entropy of δ_{P_1} and $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$ is zero, but the value

$$\begin{aligned} h(\delta_{P_1}) - \frac{t}{2}\int \log|\det(Df(z))|\, d\delta_{P_1}(z) \\ = h\left(\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}\right) - \frac{t}{2}\int \log|\det(Df(z))|\, d\left(\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}\right)(z) \\ = -\frac{t}{2}\log 32 = -t\,\frac{5}{2}\log 2 \end{aligned} \tag{7}$$

is larger than $h(v) - \frac{t}{2}\int \log|\det(Df(z))|\, dv(z)$ for any other $v \in M(f)$ (remember that f has a gap). This two linear functions $\log 4 - t\log 2$ and $-t\,\frac{5}{2}\log 2$ are equal at the value $t = -\frac{4}{3}$.

The pressure $P(t)$, therefore, is equal to

$$P(t) = \begin{cases} \log 4 - t\ \log 2 & \text{for } t \geq -\dfrac{4}{3}, \\ -t\ \dfrac{5}{2}\ \log 2 & \text{for } t < -\dfrac{4}{3}. \end{cases} \tag{8}$$

This shows the main result, because for $t = -\frac{4}{3}$ we have three equilibrium states (see Remark 3).

DEFINITION 2. Consider now $V(t) = \frac{P(-t)}{\log 4}$, $t \in \mathbf{R}$, the normalized pressure.

The function $V(t)$ therefore is equal to

$$P(t) = \begin{cases} 1 - \frac{1}{2}t & \text{for } t \geq -\frac{4}{3} \\ -t\,\frac{5}{4} & \text{for } t \leq -\frac{4}{3} \end{cases} \tag{9}$$

We showed in [1] that $V(t)$ is related to the generalized dimension. We will explain this relation more carefully in §3.

In general, for hyperbolic maps, the function $P(t)$ (or $V(t)$) is not linear, but it is, as we said before, differentiable (even real analytic on t). For rational maps, for instance, just the maps of the kind z^d, $d \in \mathbf{N}$ has pressure function linear and are hyperbolic (see [20]).

We will show now that the measures v that are invariant for f and have support in the boundary of the Deltoid region (and are not Dirac-Deltas in P_1, P_2 and P_3) satisfy the inequality:

$$2.3104 = P(-\frac{4}{3}) > h(v) - (-\frac{4}{6}) \int \log |\det(Df(z)|\, dv(z).$$

Therefore there are no more equilibrium states that are not the ones already mentioned before.

The proof is based on estimates we obtained from plots taken in a work-station.

CLAIM. – The Liapunov number of measures v as above satisfy

$$\int \log |det(Df(z))|\, dv(z) < 2.3965.$$

The maximal possible value of the entropy of v is $\log 2$ (f in the boundary is two to one), therefore

$$h(v) + (\frac{4}{3})(\frac{1}{2}) \int \log |\det(Df(z))|\, dv(z) \leq \log 2 + \frac{2}{3}(2.3965) = 2.2908 < 2.3104 = P(-\frac{4}{3}).$$

Therefore the above claim is enough to show the result we want to obtain.

The claim will follow from the fact that any periodic orbit of f in the boundary (up to the points P_1, P_2 and P_3) has Liapunov number smaller than (2.3965) and any invariant measure can be approximated by sum of measures with equal mass in points of period orbits (see [1] for references).

The boundary of the Deltoid region is given in parametric form by $2e^{2\pi iu} + e^{-4\pi iu} = g(u)$ with $u \in [0, 1]$.

For each such u we denote $A(u)$ the Jacobian determinant $Df(g(u))$.

For each value u we also denote $t(u)$ the expansion of the unit tangent vector to the curve (the boundary of the deltoid region) by the linear map $Df(g(u))$.

From the plot of the graph of $S(u) = \frac{A(u)}{t(u)}$, we can easily see that:

$$4 \leq S(u) \leq 8 \quad \text{for} \quad u \in [0, 0.1],$$
$$0 \leq S(u) \leq 5 \quad \text{for} \quad u \in [0.1, 0.9],$$

and

$$4 \leq S(u) \leq 8 \quad \text{for} \quad u \in [0.9, 1].$$

The change of coordinates h restricted to the boundary conjugates f on the Deltoid curve and d(x) (the map 2 x (mod 1) in [0, 1]). For example, $\frac{1}{3}$ and $\frac{2}{3}$ are mapped by h on P_2 and P_3.

using some kind of information in the limit when N goes to ∞. That is the reason for not existing limits on N in our context.

The point of view used by physicist mentioned above is based on the "observables" of the finite partitions. The concepts in §3 will be related to this point of view. The value t plays here the role of an external temperature (see [11, 13] for more detailed considerations).

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$$h(v) - \frac{t}{2}\int \log|\det(Df(z))|\, dv(z)$$

for any other $v \in M(f)$.

In the other way (see [1], Prop. 4) when t is very negative, then, the entropy of δ_{P_1} and $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$ is zero, but the value

$$\begin{aligned} h(\delta_{P_1}) - \frac{t}{2}\int \log|\det(Df(z))|\, d\delta_{P_1}(z) \\ = h\left(\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}\right) - \frac{t}{2}\int \log|\det(Df(z))|\, d\left(\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}\right)(z) \\ = -\frac{t}{2}\log 32 = -t\,\frac{5}{2}\log 2 \end{aligned} \tag{7}$$

is larger than $h(v) - \frac{t}{2}\int \log|\det(Df(z))|\, dv(z)$ for any other $v \in M(f)$ (remember that f has a gap). This two linear functions $\log 4 - t\log 2$ and $-t\,\frac{5}{2}\log 2$ are equal at the value $t = -\frac{4}{3}$.

The pressure $P(t)$, therefore, is equal to

$$P(t) = \begin{cases} \log 4 - t\ \log 2 & \text{for } t \geq -\dfrac{4}{3}, \\ -t\ \dfrac{5}{2}\ \log 2 & \text{for } t < -\dfrac{4}{3}. \end{cases} \tag{8}$$

This shows the main result, because for $t = -\frac{4}{3}$ we have three equilibrium states (see Remark 3).

DEFINITION 2. Consider now $V(t) = \frac{P(-t)}{\log 4}$, $t \in \mathbf{R}$, the normalized pressure.

As h is differentiable and $h'(x) \neq 0$ (up to the points 0, $\frac{1}{3}$ and $\frac{2}{3}$), we know that any periodic point p of f with period n satisfies

$$\frac{1}{n} \log |Df^n(p)(\tilde{v})| = \log 2$$

where $\tilde{v}$ is the unit tangent vector to the boundary of the Deltoid curve at p.

Note that $\frac{1}{n} \log |Df^n(p)(\tilde{v})| = \frac{1}{n} \sum_{j=0}^{n-1} \log t(f^j(p))$.

We want to estimate $\frac{1}{n} \sum_{j=0}^{n-1} A(f^j(p))$ using the function S(u) defined before.

It is easy to see from the graph of the iterate of order n of $d(x)$ that the periodic orbits have a uniform distribution on [0, 1].

Therefore there exist around 0.2 n elements of a periodic orbit of period n in the set $A = [0, 0.1] \cup [0.9, 1]$, and around 0.8 n elements in the set $B = [0.1, 0.9]$.

Now from the properties of $S(u)$ mentioned before we have

$$\begin{aligned}
&\frac{1}{n} \sum_{j=0}^{n-1} \log(A(f^j(p))) \\
&\quad = \frac{1}{n} \sum_A \log A(f^j(p)) + \frac{1}{n} \sum_B \log A(f^j(p)) \leq \frac{1}{n} \sum_A \log 8t(f^j(p)) + \frac{1}{n} \sum_B \log 5t(f^j(p)) \\
&\quad = \frac{1}{n} \sum_{j=0}^{n-1} \log t(f^j(p)) + \frac{1}{n} \sum_A \log 8 + \frac{1}{n} \sum_B \log 5 \approx \log 2 + \frac{1}{n} 0.2n \log 8 + \frac{1}{n} 0.8 \log 5 \\
&\quad = \log 2 + 0.2 \log 8 + 0.8 \log 5 = 2.3965.
\end{aligned}$$

This is the end of the proof of the Claim.

The conclusion is that there is no more equilibrium measures for the Pressure at the value $-\frac{4}{3}$ than the ones mentioned before.

3. THE SETTING OF GENERALIZED DIMENSION

For each value of $q \in \mathbf{R}$ and $\xi > 0$, consider a covering of the Deltoid region with balls of center x and radius ξ. Choose a covering with the smallest cardinality. Denote such finite covering by $\bigcup_{i \in I_\xi} B(x_\xi^i, \xi)$ where x_ξ^i are the centers of the balls and I_ξ is a set of indices with finite cardinality.

We denote $B(x, \xi)$, the ball of center x and radius ξ.

Consider now

$$\lim_{\xi \to 0} \frac{\log \sum_{i \in I_\xi} \mu(B(x_\xi^i, \xi))^q}{-\log \xi}. \tag{10}$$

This value is by definition the function $\tau(q)$. Note that $\tau(0) =$ capacity of Δ.

Capacity dimension (or box-counting dimension) and Hausdorff dimension are not always equal. The capacity dimension is the natural concept related with generalized dimension. In the setting of pressure, Hausdorff dimension is more natural (see [1]).

DEFINITION 3. The generalized dimension for the value $q \in \mathbf{R}$ is, by definition

$$D(q) = \frac{\tau(q)}{q - 1} \tag{11}$$

We point out that we are being not very rigorous with this definition because we don't know that such limit $\tau(q)$ exists. We refer the reader willing a rigorous definition to [1]. In [1], it is shown that when μ is the maximal measure then $V(\tau(q)) = q$ (see Definition 2 in §2).

From this formula and (9) in §2, we discover the expression of $\tau(q)$. It is linear by parts with a lack of differentiability at $q = \frac{5}{3}$. Therefore, the same happens for $D(q)$.

To confirm this result we will proceed as in [21], where the real Chebyshev polynomial $2x^2 - 1$ was considered, and shows that, indeed, $\tau(q)$ has this lack of differentiability at $q = \frac{5}{3}$.

REMARK 4. In order to compute $\tau(q)$, we will consider small infinitesimal triangles with area of order ξ^2. These triangles are obtained from the Markov partition associated with the change of coordinates h from the map F_2 acting on an equilateral triangle (see [7], p. 402). These triangles are more natural to use in our situation.

First of all we point out that as μ has a density with respect to Lebesgue measure, then for points inside the Deltoid region the μ-measure of a ball of radius ξ is of order ξ^2. As the capacity dimension of the Deltoid is 2, we have that the number of such balls grows like ξ^{-2}. Therefore, the contribution of these balls (or triangles) in the summation $\sum_{i \in I_\xi} \mu(B(x_\xi^i, \xi))^q$ is of order $\xi^{-2}\xi^{2q}$.

Now we have to analyze the balls (we will consider triangles) that are close to the points P_1, P_2, and P_3. We will analyze the point P_1 because the argument for the other two points P_2 and P_3 follows from the same reasoning using f^2 instead of f.

Consider an initial infinitesimal small triangle A (see Remark 4) with vertex in P_1 and contained in the Deltoid region Δ. Denote $L(A)$ the area of A and $\mu(A)$ the μ-measure of A.

For such fixed triangle A, and $n \in \mathbf{N}$, there exists a unique triangle A_n such that $f^n(A_n) = A$ and A_n has vertex in P_1 (this can be seen using the change of coordinates h and analyzing F_2 ([7], p. 404). According to [7], this region A_n has μ-measure $\mu(A_n) \approx \mu(A)4^{-n}$. As the determinant Jacobian of f in P_1 is equal to 32, we have that the area of such triangle A_n is $L(A_n) \approx L(A)32^{-n}$.

Suppose now that we want to find α such that $\mu(B(P_1, \xi)) \approx \xi^\alpha$.

REMARK 5. More rigorously speaking $\approx$ means

$$\lim_{\xi \to 0} \frac{\log \mu(B(P_1, \xi))}{\log \xi} = \alpha \qquad \text{(See [16])}$$

For simplification of the argument consider $\xi \approx (32^{-n})^{1/2}$. Then it follows easily that $\alpha = \frac{4}{5}$, because

$$\xi^{4/5} \approx (32^{-n})^{\frac{1}{2}\bullet\frac{4}{5}} = (2^{-n})^{\frac{5}{2}\frac{4}{5}} = (2^{-n})^2 = (4^{-n}) \approx \mu(B(P_1, \xi)).$$

Remember that $L(A_n) = L(A)32^{-n} = L(A)\xi^2$ and constants do not interfere in the argument (see Remark 5). Therefore, the contribution in the sum

$$\sum_{i \in I_\xi} \mu(B(x_\xi^i, \xi))^q \qquad (12)$$

of these triangles (or balls) is of order $\xi^{\frac{4}{5}q}$.

Finally, we have to analyze the triangles close to the Deltoid curve. This curve has capacity dimension one. With the same argument about Jacobian and μ-measure of triangles, one can easily show that in this case the contribution in the sum $\sum_{i \in I_\xi} \mu(B(x_\xi^i, \xi))^q$ is of order $\xi^{-1} \bullet \xi^{2q}$.

Therefore, we conclude that $\sum_{i \in I_\xi} \mu(B(x_\xi^i, \xi))^q$ is of order

$$\xi^{-2+2q} + \xi^{\frac{4}{5}q} + \xi^{-1+2q}. \qquad (13)$$

Considering now the upper envelope of the three linear maps $2 - 2q$, $-\frac{4}{5}q$ and $1 - 2q$, we conclude that

$$\tau(q) = \begin{cases} 2 - 2q & q < \frac{5}{3}, \\ -\frac{4}{5}q & q \geq \frac{5}{3}. \end{cases} \tag{14}$$

This result was already been obtained from (9) and the relation

$$V(\tau(q)) = q.$$

This is the end of the considerations about the generalized dimension of the maximal measure μ.

4. ANOTHER EXAMPLE OF TRIPLE TRANSITION: THE RESTRICTION TO THE BOUNDARY

We will consider now another example where we also have triple point transition. This will be obtained from the action of f in the boundary of the Deltoid region, but we will consider now a different potential instead of $\frac{1}{2}\log|\det(Df(z))|$.

The potential to consider the Pressure will be logarithm of the derivative of f in the direction tangent to the Deltoid curve.

As usual we will consider t an external parameter and consider $P(t)$ the Pressure of t times the above mentioned potential.

There is another measure that can also be analyzed with the above procedure related to the map f. Consider the measure ν as the pull-back by h of the arc length in the boundary of the equilateral triangle shown in Fig. 2, p. 402 in [7]. This measure has a density on the Deltoid curve $\delta\Delta$ with singularities in the points P_1, P_2 and P_3.

The map f restricted to the curve has degree 2. In fact, it is also conjugated to a shift in two symbols. This can be easily seen from the action of F_2 in the boundary of the triangles [7]. It can also be shown that the Liapunov number (there is only one now) is $\log 2$ for any point but P_1, P_2, and P_3. The reason for this fact is the same as before: F_2 has derivative equal 2 in the direction tangent to the triangle (up to the three corner points) and the change of coordinates h is differentiable.

The measure ν is the measure of maximal entropy for the action of f in the Deltoid curve.

The Liapunov number of P_1, P_2 and P_3 is $\log 4$. This can be seen from the fact that the partial derivative of f in the direction (1,0) in the point P_1 is 4. It is also true that the partial derivative in the direction $e^{\frac{2\pi i}{3}}$ in the point P_2 and the partial derivative in the direction $e^{-\frac{2\pi i}{3}}$ in the point P_3 are equal to 4. Therefore, the action of f in the Deltoid curve $\delta\Delta$ has a gap (see Def. 22 in [1]).

Analogous definitions and results as in §2 and §3 can be also obtained in this case.

Using the same arguments as before, one can show that

$$P(t) = \begin{cases} -t\log 4 & t < -1 \\ \log 2 - t\log 2 & -1 \leq t. \end{cases} \tag{15}$$

Therefore there exists a triple point transition at $t = -1$ and ν, δ_{P_1}, and $\frac{1}{2}(\delta_{P_2} + \delta_{P_3})$ coexist as equilibrium measures at $t = -1$.

From the relation

$$P(-\tau(q)) = q\log 2$$

one can also obtain $\tau(q)$ for such measure ν under the action of f in the Deltoid curve $\delta\Delta$.

This concludes the considerations about triple points.

5. A CONJECTURE ABOUT THE BIFURCATION SET OF PARAMETERS AND YANG-LEE ZEROS

The appearance of a triple point transition was shown in the previous paragraphs. This property is very much related with the dynamics of $f(z) = z^2 - 2\bar{z}$. In statistical mechanics it is well known that the analogous physical phenomena happens due to the existence of an external applied magnetic field. Therefore, using the same reasoning, we can imagine that the dynamics of $f(z) = z^2 - 2\bar{z}$ has the effect of an external applied magnetic field in our considerations. One should think about this applied field as given by an external real parameter s independent of the real parameter t.

If one supposes that the parameter s is allowed to have range in the complex plane, then an interesting phenomena happens in some concrete physical examples related to triple point transition. Suppose s_o is the real parameter value where the applied field associated with s_o has for some distinguished value t_o of the parameter, a triple point where three phases coexist. Emerging from the triple point s_o, there are three curves in the complex plane where two phases coexist (see [8]). These curves are the so called Yang-Lee zeros. The locus of the point of these three curves intercept at s_o. The locus of points of two of these three curves are not in the real line. Therefore, we can say that the Yang-Lee zeros accumulate in the real line exactly on the triple point s_o. We refer the reader to [8] for general considerations about Yang-Lee zeros. In particular, we are interested in the relation of the situation covered here with Fig. 4(a), p. 313 [8].

The two curves mentioned above are in the boundary of the set of parameters s where there exists no phase-transition for any value of the temperature t. We refer the reader also to ([22], pp. 139–141) where a concise introduction to Yang-Lee zeros is presented.

As it is mentioned in [22], we point out that the values of s that are not real, have no physical meaning, but they are important in the theoretical understanding of the problem of triple point phase transition.

Trying to explore the analogy of the dynamics of polynomials with the concrete physical problem a little further, we can reason in the following way: the triple point parameter s_o is equal to 2 in our situation and the family $f_s(z) = z^2 - s\bar{z}$ represents other possible applied fields.

It is well known that for real polynomials on $\mathbf{R}$, of the form $g(x) = x^2 - sx$, where $s \in \mathbf{R}$ and $x \in \mathbf{R}$, if the critical point $\frac{s}{2}$ goes to ∞ under iterations of g, then the map g is hyperbolic and equilibrium measures are always unique. Therefore, there exists no phase transition.

For maps of the form

$$f_s(z) = z^2 - s\bar{z}, \quad \text{where} \quad s \in \mathbf{C}$$

and $z \in \mathbf{C}$, the critical set (that is, the set of points z where the determinant Jacobian of f_s is zero) is the circle of center zero and radius $\frac{\|s\|}{2}$.

For large values of s (that is, when $\|s\|$ is large), the critical set goes to ∞ under iterations of f_s. The nonwandering set has a Cantor set structure. The value $s_o = 2$ is one of the bifurcation values of the parameter s. That means that for $s > 2$, the critical set goes to ∞, and for $s < 2$, there exist points in the critical set that do not go to ∞ under iterations of f_s.

For some values of $s < 2$, there seems to exist an f_s-invariant measure with a density with respect to two-dimensional Lebesgue measure. All these properties can be seen by plotting pictures in a computer. We use a computer to analyze the set of all bifurcation values of the parameter $s \in \mathbf{C}$ (coming from ∞). The locus of these bifurcation values are in a cardioid curve with extremes in -6 and 2, as shown in Fig. 2.

For the values s on the cardioid curve, there exist always points a arbitrarily close to s, such that the critical set of f_a goes to ∞, and also points b arbitrarily close to s, such that there exist points of the critical set of f_b that do not go to ∞.

In a local neighborhood of $s_o = 2$ in $\mathbf{C}$, there exist two curves in the cardioid emerging from the point s_o. These two curves are in the boundary of the set of values of s where the critical point of f_s goes to ∞ under iterations of f_s.

In a forthcoming paper (see [25]), we hope to analyze the rigorous mathematical proof of the following conjecture: For points s outside the interior part of the cardioid given in Fig. 2, the variational problem

$$\sup_{v \in M(f_s)} \left\{ h(v) - \frac{t}{2} \int \log |\det(Df_s(z))| \, dv(z) \right\}$$

has a unique equilibrium measure for any value of t. Therefore, there is no phase transition for such applied field s.

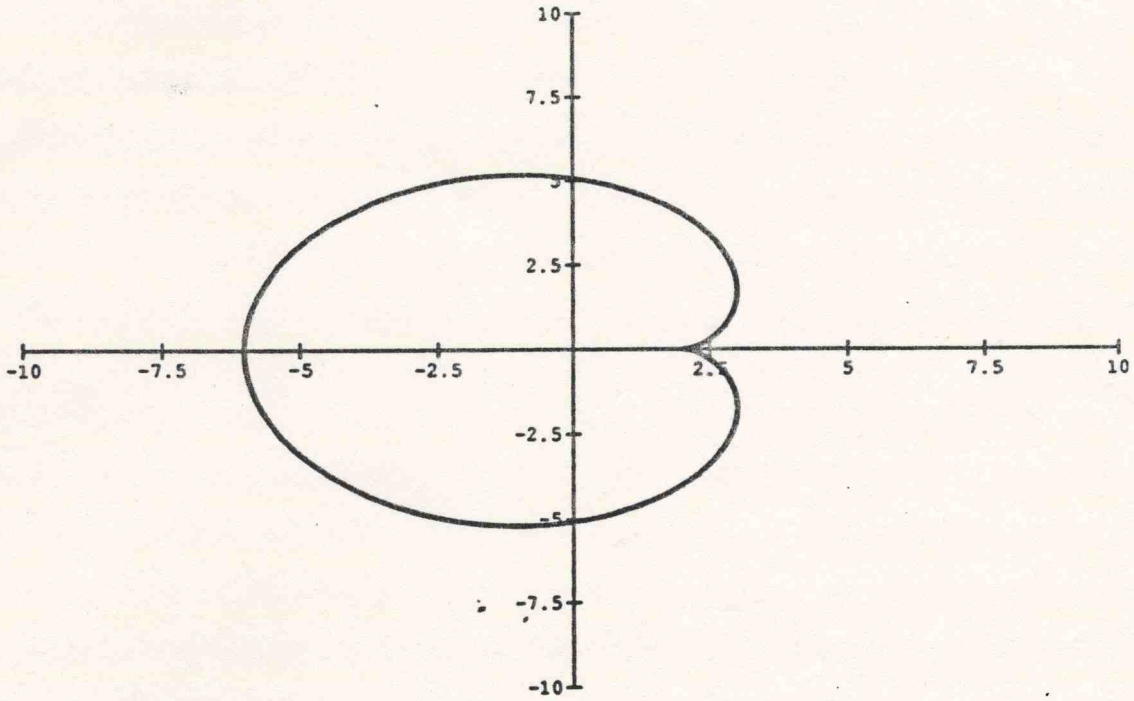

Figure 2.

We also conjecture that for each point s in the cardioid curve, more than one equilibrium measure can exist for some distinguished values of t (can depend on s).

We believe that the analogy with the concrete physical Yang-Lee zeros of a triple point transition as mentioned in the beginning of §4 is transparent.

We point out a similar and well known result in the literature [22], the bifurcation parameter set for the family

$$v_s(z) = z^2 + s, \quad s \in \mathbf{C}, \quad z \in \mathbf{C}.$$

In this case the bifurcation set is known as the Mandelbrot set [22].

We believe it is worthwhile of a deeper investigation of the bifurcation set of the family f_s.

For some values of s close to zero, it seems that equilibrium measures are also unique for any value of t. The nonwandering set of such f_s appears to have a fractal nature (see [25]).

6. CONCLUSION

The phenomena of triple point transition presented in the Potts Model, can be analyzed by means of the observables associated with the maximal measure μ of a real polynomial in the plane.

Increasing an external parameter q, we reach the value $q = \frac{5}{3}$ where the singularities of the measure μ at the points P_1, P_2, and P_3 have a contribution of higher order than the contribution of the points inside the Deltoid region. This is the setting of generalized dimension.

At the value $t = -\frac{4}{3}$, we have three equilibrium measures: the maximal measure μ (not magnetic) and the measures δ_{P_1} and $\frac{1}{2}\delta_{P_2} + \frac{1}{2}\delta_{P_3}$ of magnetic and anti-ferromagnetic nature. This is the setting of pressure.

If we want to be more precise, we should say that we have triple point transition when the convex simplex of the set of equilibrium states is the convex hull of three distinct and independent invariant probabilities.

We refer the reader to [23, 24] for more explanations about the phenomena of phase transition and its relation with real problems in physics.

In terms of spin-lattices, the model is related to a semi-infinite lattice N with four possible spin-components in each sitte of the lattice. Until we reach the transition value, any one of the four spin-components has equal probability and there is no magnetization. After the transition parameter is reached, two magnetic and anti-ferromagnetic equilibrium measures coexist. The fourth spin-component has no magnetization in the critical parameter and also after this parameter is attained (see Remark 3).

This can be considered as an anisotropy in some spin-component directions.

The above considerations shows an analogy with the Potts Model and the phenomena of triple point phase transition. (See [25] for more results in this direction).

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