Selection of measure and a Large Deviation Principle for the general one-dimensional XY model

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Abstract

We consider (M,d) a connected and compact manifold and we denote by X the Bernoulli space $M^{\mathbb{N}}$. The shift acting on X is denoted by σ.

We analyze the general XY model, as presented in a recent paper by A. T. Baraviera, L. M. Cioletti, A. O. Lopes, J. Mohr and R. R. Souza. Denote the Gibbs measure by $\mu_c := h_c \nu_c$, where h_c is the eigenfunction, and, ν_c is the eigenmeasure of the Ruelle operator associated to cf. We are going to prove that any measure selected by μ_c, as $c \to +\infty$, is a maximizing measure for f. We also show, when the maximizing probability measure is unique, that it is true a Large Deviation Principle, with the deviation function $R_+^\infty = \sum_{j=0}^\infty R_+(\sigma^j)$, where $R_+ := \beta(f) + V\circ\sigma - V - f$, and, V is any calibrated subaction.

1 Introduction

We consider (M,d) a connected and compact manifold and we denote by X the Bernoulli space $M^{\mathbb{N}}$. The shift acting on X is denoted by σ.

We point out that the number of preimages by σ of each point is not countable.

Let $f : X \to \mathbb{R}$ be a fixed Holder potential defined in the Bernoulli space X. We denote by m the Lebesgue probability on M. We suppose without

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lost of generality that the diameter of the manifold M is smaller than one. This distance induces another one, in the usual fashion, on $M^{\mathbb{N}}$ [4].

We are interested in the Gibbs state (for finite and zero temperature) associated to the potential f. This model is called the general XY model in [4]. We refer the reader to such work for a detailed explanation about the motivation for considering such kind of problems. We point out that in the literature in Physics what is called the XY model is the case when $M = S^1$, and, the potential depends on a finite number of coordinates. In [4] and here the hypothesis are more general.

Classical references in the XY model are [15], [24] and [28]. A nice reference for general results in Statistical Mechanics is [13].

In order to define a transfer operator we need a probability a priori on M which we will denote by $d\,m$. In the case $M = S^1$ it usually consider the Lebesgue measure dx [28].

First we will recall some definitions and results from [4].

Definition 1. *Let $\mathcal{C}$ be the space of continuous functions from $X = M^{\mathbb{N}}$ to $\mathbb{R}$. We define the Ruelle operator on $\mathcal{C}$, associated to the Holder potential $f : M^{\mathbb{N}} \to \mathbb{R}$, which is the linear operator that gets $w \in \mathcal{C}$, and sends to $L_f(w) \in \mathcal{C}$, defined for any $x = (x_0, x_1, x_2, ...) \in X$, by*

$$L_f(w)(x) = \int e^{f(ax)}\, w(ax)\, dm\,(a)\,,$$

where ax represents the sequence $(a, x_0, x_1, x_2, ...) \in X$, and $dm\,(a)$ is the Lebesgue probability on M.

Following [4], for a real value c we consider β_c the eigenvalue, h_c the eigenfunction, and $g_c = cf + \log(h_c) - \log(h_c \circ \sigma) - \log(\beta_c)$ the normalized function associated to the Ruelle operator $L_{c\,f}$ obtained from cf. We also denote ν_c the eigenmeasure of $L^*_{c\,f}$, and, $\mu_c := h_c \nu_c$, the Gibbs probability of the potential cf.

As usual, by notation $f^n(x) = \sum_{j=0}^{n-1} f(\sigma^j(x))$, for any $n \in \mathbb{N}$, $x \in X$.

Remark on notation: the iterated Ruelle Operator $L^n_f w(x)$, $n = 1, 2, 3...$, can be written as

$$\int_{a_n...a_1} e^{f^n(a_n...a_1 x)} w(a_n...a_1 x)\, da_1...da_n, \quad or \quad \int_{\sigma^n z = x} e^{f^n(z)} w(z)\, dm.$$

We denote by $\mathcal{M}_\sigma$ the compact set of invariant probability measures for σ.

We consider the following problem: for the given $f : X \to \mathbb{R}$, we want to find probability measures that maximize, over $\mathcal{M}_\sigma$, the value $\int f(x)\, d\mu(\mathbf{x})$.

Definition 2. *We define*

$$\beta(f) = \max_{\mu \in \mathcal{M}_\sigma} \left\{ \int f d\mu \right\} .$$

Any of the measures which attains the maximal value will be called a maximizing probability measure for f, which is sometimes denoted by μ_∞.

In Section 2 we are going to prove the following

Theorem 3. *Any weak*-limit of subsequence of μ_c, $(c \to +\infty)$, is a maximizing probability measure to f.*

In this way one can say that any convergent subsequence of Gibbs states at positive temperature selects maximizing probabilities. In this result we do not assume uniqueness of the maximizing probability.

The similar result for the Classical Thermodynamical Formalism considers the shift acting on the Bernoulli space $\{1, 2, ., d\}^{\mathbb{N}}$ [27]. In this case one can consider entropy and pressure and the proof is trivial (see [11] [10]). Here we can not take advantage of this and the proof requires other methods.

Definition 4. *A continuous function $V : X \to \mathbb{R}$ is called a* calibrated subaction *for $f : X \to \mathbb{R}$, if, for any $y \in X$, we have*

$$V(y) = \max_{\sigma(x)=y} [f(x) + V(x) - \beta(f)]. \tag{1}$$

This can be also be expressed as

$$\beta(f) = \max_{a \in M} \{f(ay) + V(ay) - V(y)\}.$$

One can show that for any x in the support of the maximizing probability measure for f we have that

$$V(\sigma(x)) - V(x) - f(x) + \beta(f) = 0.$$

In this way if we know the value $\beta(A)$, then a calibrated subaction V for f helps to identify the union of the supports of maximizing probabilities μ_∞ for f. The above equation can be eventually true outside the union of the supports of the maximizing probabilities μ.

If the maximizing probability is unique, then the calibrated subaction is unique up to an additive constant [3] [4] [12].

It is known [4] that $\frac{1}{c}\log(h_c)$, $c \in \mathbb{R}$, is a equicontinuous family. Any limit of subsequence $V = \lim_{n\to\infty} \frac{1}{c_n}\log(h_{c_n})$, $c_n \to \infty$, is a calibrated subaction [4].

We denote in the following $R_+^\infty = \sum_{j=0}^\infty R_+(\sigma^j)$, with $R_+ := \beta(f) + V \circ \sigma - V - f$, where V is any calibrated subaction.

We say that the potential A depends on two variables if for any $x = (x_0, x_1, x_2, x_3....) \in X$ we have that the value $A(x_0, x_1, x_2, x_3....)$ is independent of $(x_2, x_3, x_4, ...)$. This case is also known as the "nearest-neighbour" interaction".

The next theorem was shown to be true in the case the potential A depends on two coordinates in [22].

In section 3 we consider the case where the maximizing measure for f is unique, and prove that the family μ_c satisfies the following Large Deviation Principle:

Theorem 5. *Suppose the maximizing probability for f is unique. Then, for any closed set F, and any open set A:*

$$\limsup_{c\to\infty} \frac{1}{c}\log(\mu_c(F)) \leq -\inf_{x\in F} R_+^\infty(x),$$

$$\liminf_{c\to\infty} \frac{1}{c}\log(\mu_c(A)) \geq -\inf_{x\in A} R_+^\infty(x).$$

The function R_+^∞ is lower semicontinuous, and can attains the value ∞ in some points.

The above theorem will be a consequence of a more general result:

Theorem 6. *Suppose the maximizing probability for f is unique. Consider any point $x \in X$, then, for any closed set F, and any open set A:*

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^n \chi_F)(x)) = -\inf_{z\in F}(R_+^\infty(z)),$$

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^n \chi_A)(x)) = -\inf_{z\in A}(R_+^\infty(z)).$$

Theorem 5 is a consequence of the above just by taking first $n \to \infty$ [27], and, then, making $c \to \infty$

We point out that the reasoning which proves this last result can also be applied to the Classical Thermodynamic Formalism setting [27], where the

Bernoulli space is $\{1, 2, ..., d\}^{\mathbb{N}}$, to get the analogous result. This proof of the L. D. P. does not use the involution kernel as in [3].

In [21] is presented another kind of Large Deviation Principle: the setting of zeta measures. In this case the proof do not require that the maximizing probability is unique.

2 The selection of measure

Lemma 7. *Let V be a calibrated subaction, such that, $V = \lim_{c\to\infty} \frac{1}{c} \log(h_c)$, and, $R_- = f + V - V \circ \sigma - \beta(f)$, which is the limit function of the g_c/c associated. For each $\epsilon > 0$ there exists a constant ψ_ϵ such that for any $x \in X$*

$$m(\{a \in M : R_-(ax) > -\epsilon\}) > \psi_\epsilon > 0.$$

Proof. Suppose g_c converges to R_-, and then write $g_c = cR_- + \delta_c$, where $|\delta_c|_\infty/c \to 0$. Using that V is a calibrated subaction, we have $R_- \leq 0$.

We fix $\epsilon > 0$, and we define

$$A_\epsilon := \{a : R_-(ax) \leq -\epsilon\}$$

$$B_\epsilon := \{a : R_-(ax) > -\epsilon\}.$$

V is Holder, so R_- is Holder, then, it is a continuous function on the first symbol. In this way, A_ϵ and B_ϵ are mensurable sets. We have:

$$1 = L_{g_c}1(x) = \int e^{g_c(ax)}da = \int e^{cR_-(ax)+\delta_c(ax)}da.$$

Therefore,

$$\begin{aligned}
1 &= \int_{A_\epsilon} e^{cR_-(ax)+\delta_c(ax)}da + \int_{B_\epsilon} e^{cR_-(ax)+\delta_c(ax)}da \\
&\leq \int_{A_\epsilon} e^{-c\epsilon+\delta_c(ax)}da + \int_{B_\epsilon} e^{0+\delta_c(ax)}da \\
&\leq \int_{A_\epsilon} e^{-c\epsilon+|\delta_c|_\infty}da + \int_{B_\epsilon} e^{0+|\delta_c|_\infty}da \\
&= e^{-c\epsilon+|\delta_c|_\infty}m(A_\epsilon) + e^{|\delta_c|_\infty}m(B_\epsilon) \\
&\leq e^{-c\epsilon+|\delta_c|_\infty} + e^{|\delta_c|_\infty}m(B_\epsilon).
\end{aligned}$$

Let $c_0 > 0$ be such that $e^{-c_0\epsilon+|\delta_{c_0}|_\infty} \leq 1/2$. Then, it follows that

$$1/2 \leq e^{|\delta_{c_0}|_\infty}m(B_\epsilon),$$

so,

$$m(B_\epsilon) \geq \frac{1}{2e^{|\delta_{c_0}|_\infty}}.$$

Then, we just take $\psi_\epsilon = \frac{1}{3e^{|\delta_{c_0}|_\infty}}$, and the result follows. $\square$

Proof of Theorem 3

Proof. Let ν be an accumulation point of μ_c given by a certain subsequence $c_j \to \infty$. Let c_i be a subsequence of this one such that there exists the limit V of the sequence $\frac{1}{c_i}\log h_{c_i}$. Let $R_- := f + V - V \circ \sigma - \beta(f)$ be the function associated to such limit. Then, $g_{c_i}/c_i \to R_-$. Define $a := \lim_{c_i \to \infty} \mu_{c_i}(R_-) = \lim_{c_j \to \infty} \mu_{c_j}(R_-)$. Then, it follows that $a \leq 0$. We are going to show that $a \geq 0$. More precisely we are going to show that for any fixed $x \in X$:

$$\liminf_{i,n \to \infty} L^n_{g_{c_i}}(R_-)(x) \geq 0.$$

We write $\log h_{c_i} = c_i V + \delta_i$, where

$$\frac{|\delta_i|_\infty}{c_i} \to 0.$$

We also write

$$L^n_{g_{c_i}}(R_-)(x) = \int_{a_n \ldots a_1} e^{g^n_{c_i}(a_n \ldots a_1 x)} R_-(a_n \ldots . a_1 x) da_1 \ldots d_{a_n}$$

$$= \frac{\int_{a_n \ldots a_1} e^{c_i f^n(a_n \ldots a_1 x) + \log(h_{c_i}(a_n \ldots a_1 x)) - \log(h_{c_i}(x))} R_-(a_n \ldots a_1 x) da_1 \ldots da_n}{\int_{a_n \ldots a_1} e^{c_i f^n(a_n \ldots a_1 x) + \log(h_{c_i}(a_n \ldots a_1 x)) - \log(h_{c_i}(x))} da_1 \ldots da_n}$$

$$= \frac{\int_{a_n \ldots a_1} e^{c_i f^n(a_n \ldots a_1 x) + \log(h_{c_i}(a_n \ldots a_1 x))} R_-(a_n \ldots a_1 x) da_1 \ldots da_n}{\int_{a_n \ldots a_1} e^{c_i f^n(a_n \ldots a_1 x) + \log(h_{c_i}(a_n \ldots a_1 x))} da_1 \ldots da_n}$$

$$= \frac{\int_{a_n \ldots a_1} e^{c_i f^n(a_n \ldots a_1 x) + c_i V(a_n \ldots a_1 x) + \delta_i(a_n \ldots a_1 x) - c_i V(x) - c_i \beta(f)} R_-(a_n \ldots a_1 x) da_1 \ldots da_n}{\int_{a_n \ldots a_1} e^{c_i f^n(a_n \ldots a_1 x) + c_i V(a_n \ldots a_1 x) + \delta_i(a_n \ldots a_1 x) - c_i V(x) - c_i \beta(f)} da_1 \ldots da_n}$$

$$= \frac{\int_{a_n \ldots a_1} e^{c_i R^n_-(a_n \ldots a_1 x) + \delta_i(a_n \ldots a_1 x)} R_-(a_n \ldots a_1 x) da_1 \ldots da_n}{\int_{a_n \ldots a_1 x} e^{c_i R^n_-(a_n \ldots a_1 x) + \delta_i(a_n \ldots a_1 x)} da_1 \ldots da_n}.$$

For a fixed $\varepsilon > 0$, we define the sets:

$$A_n := \{a_n \ldots a_1 : R_-(a_n \ldots a_1 x) < -\varepsilon\},$$
$$B_n := \{a_n \ldots a_1 : R_-(a_n \ldots a_1 x) \geq -\varepsilon\},$$
$$C_n := \{a_n \ldots a_1 : R_-(a_n \ldots a_1 x) > -\frac{\epsilon}{2}\}.$$

Clearly, we have that $C_n \subseteq B_n$.

As V is a calibrated subaction, then C_n is not empty. We remark that $A_n \cup B_n = [0,1]^n$, and, by the Lemma above, for each $a_{n-1}...a_1$, we have that $m\{a_n : a_n...a_1 \in C_n\} > \psi_\epsilon > 0$.

Then:

$$\int_{A_n} e^{c_i R_-^n + \delta_i} da_1...da_n = \int_{A_n} e^{c_i R_- + \delta_i} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$\leq \int_{A_n} e^{-c_i \varepsilon + |\delta_i|_\infty} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$= e^{-c_i \varepsilon + |\delta_i|_\infty} \int_{A_n} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$\leq e^{-c_i \varepsilon + |\delta_i|_\infty} \int_{M^n} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$= e^{-c_i \varepsilon + |\delta_i|_\infty} \int_{M^{n-1}} e^{c_i R_-^{n-1}} da_1...da_n,$$

and,

$$\int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n = \int_{B_n} e^{c_i R_- + \delta_i} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$\geq \int_{C_n} e^{c_i R_- + \delta_i} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$\geq \int_{C_n} e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$\geq e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \int_{C_n} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$\geq e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \int_{M^{n-1}} \int_{\{a_n : a_n...a_1 \in C_n\}} e^{c_i R_-^{n-1} \circ \sigma} da_1...da_n$$

$$= e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \int_{M^{n-1}} e^{c_i R_-^{n-1}} \int_{\{a_n : a_n...a_1 \in C_n\}} da_1...da_n$$

$$= e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \psi_{\epsilon/2} \int_{M^{n-1}} e^{c_i R_-^{n-1}} da_1...da_n.$$

It follows that

$$0 \leq \liminf_{i,n \to \infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} da_1...da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i}} da_1...da_n$$

$$\leq \limsup_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i}}{\int_{B_n} e^{c_i R_-^n + \delta_i}} da_1...da_n$$

$$\leq \limsup_{i,n\to\infty} \frac{e^{-c_i\varepsilon + |\delta_i|_\infty}}{\psi_{\epsilon/2} e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty}}$$

$$\leq \limsup_{i,n\to\infty} e^{-c_i\,\varepsilon/2 + 2|\delta_{c_i}|_\infty} \psi_{\epsilon/2}^{-1}$$

$$= \limsup_{i,n\to\infty} e^{c_i\,(-\epsilon/2 + 2\frac{|\delta_{c_i}|_\infty}{c_i})} \psi_{\epsilon/2}^{-1} = 0.$$

In the same way

$$0 \leq \liminf_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} R_- \; da_1...da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n(-\varepsilon)}$$

$$\leq \limsup_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} R_- \; da_1...da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n(-\varepsilon)}$$

$$\leq \limsup_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} da_1...da_n \,(-|R_-|_\infty)}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n(-\varepsilon)}$$

$$\leq \limsup_{i,n\to\infty} \frac{e^{-c_i\varepsilon + |\delta_i|_\infty}(-|R_-|_\infty)}{\psi_{\epsilon/2}(-\varepsilon) e^{-c_i\,\epsilon/2 - |\delta_i|_\infty}} = 0.$$

From the above, and writing $\int_{M^n} da_1...da_n = \int_{A_n} da_1...da_n + \int_{B_n} da_1...da_n$, we have:

$$\liminf_{c_i,n\to\infty} L^n_{g_{c_i}}(R_-)(x) = \liminf_{c_i,n\to\infty} \frac{\int_{a_n...a_1} e^{c_i R_-^n(a_n...a_1x) + \delta_i(a_n...a_1x)} R_-(a_n...a_1x) da_1...da_n}{\int_{a_n...a_1} e^{c_i R_-^n(a_n...a_1x) + \delta_i(a_n...a_1x)} da_1...da_n}$$

$$\geq \liminf_{c_i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} R_- da_1...da_n + \int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n(-\varepsilon)}{\int_{A_n} e^{c_i R_-^n + \delta_i} da_1...da_n + \int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n}$$

$$= \liminf_{c_i,n\to\infty} \frac{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n(-\varepsilon)}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1...da_n}$$

$$\geq -\varepsilon.$$

Taking $\varepsilon \to 0$, we get our claim. $\square$

This ends the proof of our first main result.

The bottom line is: a convergent subsequence of Gibbs states at positive temperature selects maximizing probabilities (eventually, different limits of subsequences can localize different probabilities).

3 On the Large Deviation Principle

On this section we are going to prove Theorem 5. The proof will follow from some lemmas.

We suppose that the maximizing measure for f is unique, and we denote μ_∞ the maximal one.

Under this assumption, two calibrated subactions differ by a constant. This follows from proposition 5 in [3]. In particular the function $R_+ := \beta(f) + V \circ \sigma - V - f$ is well defined. The function $R_- := -R_+$ is the unique accumulation point of g_c/c, on the uniform topology, so $g_c/c \to R_-$ uniformly.

Given a double indexed sequence $z_{c,n}$, $c \in \mathbb{R}$, $n \in \mathbb{N}$, we say that $\lim_{c,n\to\infty} z_n = w$, in for any given $\epsilon > 0$, there exists an $M > 0$, such that, if $c, n > M$, then $|z_{c,n} - w| < \epsilon$.

We are going to prove a stronger result than Theorem 5.

Theorem 8. *Fixed any point $x \in X$, for any closed set F and open set A:*

$$\limsup_{c,n\to\infty} \frac{1}{c} \log((L^n_{g_c}\chi_F)(x)) \le \sup_{z\in F} R^\infty_-(z) = -\inf_{z\in F}(R^\infty_+(z)),$$

$$\liminf_{c,n\to\infty} \frac{1}{c} \log((L^n_{g_c}\chi_A)(x)) \ge \sup_{z\in A} R^\infty_-(z) = -\inf_{z\in A}(R^\infty_+(z)).$$

The function R^∞_+ is lower semi-continuous.

Theorem 5 is a consequence of taking $n \to \infty$, and, then taking $c \to \infty$

Lemma 9. *The function R^∞_+ is lower semi-continuous.*

Proof. We take $z, z_j \in X$, with $z_j \to z$. We are going to show that

$$\liminf_{j\to\infty} R^\infty_+(z_j) \ge R^\infty_+(z).$$

In the case $R^\infty_+(z) = 0$ the result is true.

First case: $R^\infty_+(z) = \infty$.

Given $M > 0$, let n be such that $R^n_+(z) > 2M$. We fix n_1, such that, $\frac{(1-\theta)\theta^{n_1}M}{|R|_\theta} < 1$. Let n_0 be such that for $j \ge n_0$, we have $d(z_j, z) < \frac{(1-\theta)\theta^{n_1}M\theta^n}{|R|_\theta}$. Then, for $j \ge n_0$:

$$\begin{aligned} R^n_+(z_j) &\ge R^n_+(z) - |R|_\theta(d(z_j, z) + ... + d(\sigma^{n-1}(z_j), \sigma^{n-1}z)) \\ &\ge 2M - |R|_\theta(\frac{\theta^{n_1}M}{|R|_\theta}) \ge M. \end{aligned}$$

It follows that

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq M.$$

Taking $M \to +\infty$:

$$\liminf_{j\to\infty} R_+^\infty(z_j) = +\infty.$$

Second case: $R_+^\infty(z) = M > 0$.
Fixed $\varepsilon > 0$, there exist n, such that, $R_+^n(z) > M - \varepsilon/2$. Let n_0 be such that for $j \geq n_0$, we have $d(z_j, z) < \frac{(1-\theta)\theta^n\varepsilon}{2|R|_\theta}$. Then, for $j \geq n_0$:

$$R_+^n(z_j) \geq (M - \varepsilon/2) - |R|_\theta \left(\frac{\varepsilon}{2|R|_\theta} \right) = M - \varepsilon.$$

So, we have that

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq M - \varepsilon.$$

Taking $\varepsilon \to 0$, we get:

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq M.$$

□

Remark: Note that $R := R_- = -R_+$.

We note that in [4] it is proved that $\frac{1}{c}\log(\beta_c) \to \beta(f)$. We denote $\varepsilon_c = \log(\beta_c) - c\beta(f)$. Then, we have $\frac{\varepsilon_c}{c} \to 0$.

Lemma 10.

$$\lim_{c,n\to\infty} \left(\frac{1}{c}\log((L_{cR}^n 1)(x)) - \frac{n.\varepsilon_c}{c} \right) = 0,$$

in particular, for a fixed k:

$$\lim_{c,n\to\infty} \frac{1}{c}\log((L_{cR}^n 1)(x)) - \frac{1}{c}\log((L_{cR}^{n+k} 1)(x)) = 0.$$

Proof. Let a an accumulation point of $\frac{1}{c}\log((L_{cR}^n 1)(x)) - \frac{n.\varepsilon_c}{c}$, when $c, n \to \infty$. Then, there exists $c_j, n_j \to \infty$, such that,

$$\lim_{j\to\infty} \left(\frac{1}{c_j}\log((L_{c_jR}^{n_j} 1)(x)) - \frac{n_j.\varepsilon_{c_j}}{c_j} \right) = a.$$

Following [4] we can take a subsequence $\{j_i\}$ such that $\dfrac{1}{c_{j_i}}\log(h_{c_{j_i}})$ converges uniformly to a calibrated subaction V. So there exist sequences $c_i, n_i \to \infty$ such that:

$$\lim_{i\to\infty} \left(\frac{1}{c_i}\log((L_{c_iR}^{n_i} 1)(x)) - \frac{n_i.\varepsilon_{c_i}}{c_i} \right) = a, \text{ and }, \lim_{i\to\infty} \frac{1}{c_i}\log(h_{c_i}) = V.$$

Denoting $\log(h_{c_i}) = c_i V + \delta_{c_i}$ where $|\delta_{c_i}|_\infty / c_i \to 0$, we have:

$$
\begin{aligned}
0 &= \lim_{i\to\infty} \frac{1}{c_i} \log((L^{n_i}_{g_{c_i}} 1)(x)) \\
&= \lim_{i\to\infty} \frac{1}{c_i} \log(\int_{\sigma^{n_i}(z)=x} e^{c_i f^{n_i}(z) + \log(h_{c_i}(z)) - \log(h_{c_i}(x)) - n_i \log(\beta_{c_i})}\, dm) \\
&= \lim_{i\to\infty} \frac{1}{c_i} \log(\int_{\sigma^{n_i}(z)=x} e^{c_i f^{n_i}(z) + c_i V(z) - c_i V(x) - n_i c_i \beta(f) + \delta_{c_i}(z) - \delta_{c_i}(x) - n_i \varepsilon_{c_i}}\, dm) \\
&= \lim_{i\to\infty} \frac{1}{c_i} \log(\int_{\sigma^{n_i}(z)=x} e^{c_i R^{n_i}(z) + \delta_{c_i}(z) - \delta_{c_i}(x) - n_i \varepsilon_{c_i}}\, dm) \\
&= \lim_{i\to\infty} \frac{1}{c_i} \log(\int_{\sigma^{n_i}(z)=x} e^{c_i R^{n_i}(z) - n_i \varepsilon_{c_i}}\, dm) \\
&= \lim_{i\to\infty} \left(\frac{1}{c_i} \log(\int_{\sigma^{n_i}(z)=x} e^{c_i R^{n_i}(z)}\, dm) - \frac{n_i \varepsilon_{c_i}}{c_i} \right) = a.
\end{aligned}
$$

This shows that any accumulation point have to be equal to zero. □

The first inequality of Theorem 8:

Proposition 11. *For any closed set $F \subseteq X$*

$$
\limsup_{c,n\to\infty} \frac{1}{c} \log((L^n_{g_c} \chi_F)(x)) \le \sup_{z\in F} R^\infty_-(z).
$$

Proof. For a fixed k, we have that

$$
\begin{aligned}
&\limsup_{c,n\to\infty} \frac{1}{c} \log((L^{n+k}_{g_c} \chi_F)(x)) = \limsup_{c,n\to\infty} \frac{1}{c} \log(\frac{(L^{n+k}_{g_c} \chi_F)(x)}{(L^{n+k}_{g_c})(x)}) \\
&= \limsup_{c,n\to\infty} \frac{1}{c} \log(\frac{\int_{\sigma^{n+k}(z)=x} e^{cR^{n+k}_-(z)} \chi_F(z)\, dm}{\int_{\sigma^{n+k}(z)=x} e^{cR^{n+k}_-(z)}\, dm}) \\
&= \limsup_{c,n\to\infty} \frac{1}{c} \log(\frac{\int_{\sigma^{n+k}(z)=x} e^{cR^{n+k}_-(z)} \chi_F(z)\, dm}{\int_{\sigma^{n}(y)=x} e^{cR^{n}_-(y)}\, dm}) \\
&= \limsup_{c,n\to\infty} \frac{1}{c} \log(\frac{\int_{\sigma^{n}(y)=x} (e^{cR^{n}_-(y)} \int_{\sigma^k(z)=y} e^{cR^k_-(z)} \chi_F(z)\, dm)\, dm}{\int_{\sigma^{n}(y)=x} e^{cR^{n}_-(y)}\, dm}).
\end{aligned}
$$

Note, however that

$$
\int_{\sigma^k(z)=y} e^{cR^k_-(z)} \chi_F(z)\, dm \le e^{c \sup_{z\in F} R^k_-(z)},
$$

then,

$$\limsup_{c,n\to\infty}\frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \leq \limsup_{c,n\to\infty}\left(\sup_{z\in F} R^k_-(z)\right) = \sup_{z\in F} R^k_-(z).$$

For each k fixed, we have that R^k_- is a continuous function, and $F\subset X$ is a compact set, then, there exist $y_k\in F$, such that, $\sup_{z\in F} R^k_-(z) = R^k_-(y_k)$. Define

$$Y_k := \{y\in F : \limsup_{c,n\to\infty}\frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \leq R^k_-(y)\}.$$

Then Y_k is closed (because R^k_- is a continuous function) and not empty (because $y_k\in Y_k$). Using $R_-\leq 0$ we have

$$Y_1\supseteq Y_2\supseteq ...$$

These sets are closed and not empty, then there exist some $x_0\in\bigcap_{k\geq 1} Y_k$. So, for each k:

$$\limsup_{c,n\to\infty}\frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \leq R^k_-(x_0).$$

Using the fact that $R^k_-(x_0)\to R^\infty_-(x_0)$, we conclude that

$$\limsup_{c,n\to\infty}\frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \leq R^\infty_-(x_0) \leq \sup_{z\in F} R^\infty_-(z).$$

□

The second inequality of Theorem 8

Suppose that A is open. So, there exists n_0, such that, for $n\geq n_0$, and, $x\in X$, there exists $y\in A$, such that, $\sigma^n(y)=x$. More precisely, given $y=y_1y_2...$ in A, let $\epsilon>0$, such that, $B(y,\epsilon)\subset A$. Let n_0 such that $\frac{\theta^{n_0}}{1-\theta}<\epsilon$. If $z\in X$ coincide with $y_1...y_{n_0}$ in its firsts symbols, then, clearly, $d(z,y)\leq\theta^{n_0}+\theta^{n_0+1}+...\leq\frac{\theta^{n_0}}{1-\theta}<\epsilon$, and, so $z\in A$. We conclude that given $x\in X$, we have that $y_1...y_{n_0}x\in A$.

Lemma 12. *There exist $y_0\in X$ such that*

$$\liminf_{c,n\to\infty}\frac{1}{c}\log((L^n_{g_c}\chi_A)(x)) \geq \limsup_{k\to\infty}(\sup_{z\in A,\,\sigma^k(z)=y_0} R^k_-(z)).$$

Proof. For a fixed $k \geq n_0$, we have

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^{n+k}\chi_A)(x)) = \liminf_{c,n\to\infty} \frac{1}{c}\log(\frac{\int_{\sigma^n(y)=x} e^{cR_-^n(y)}(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\,dm)\,dm}{\int_{\sigma^n(y)=x} e^{cR_-^n(y)}\,dm})$$

$$\geq \liminf_{c,n\to\infty} \frac{1}{c}\log(\inf_{y\in X}\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\,dm)$$

$$= \liminf_{c\to\infty} \inf_{y\in X} \frac{1}{c}\log(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\,dm).$$

Then, we get

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^{n}\chi_A)(x)) \geq \limsup_{k\to\infty}\liminf_{c\to\infty} \inf_{y\in X} \frac{1}{c}\log(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\,dm).$$

Let $y_{c,k}$ be such that

$$\inf_{y\in X} \frac{1}{c}\log(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\,dm) > \frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR_-^k(z)}\chi_A(z)\,dm) - \frac{1}{k}.$$

As X is a compact set, let y_0 be an accumulation point of $y_{c,k}$, when $c, k \to \infty$. Then, we have:

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^{n}\chi_A)(x)) \geq \limsup_{k\to\infty}\liminf_{c\to\infty} \frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR_-^k(z)}\chi_A(z)\,dm) - \frac{1}{k}$$

$$= \limsup_{k\to\infty}\liminf_{c\to\infty} \frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR_-^k(z)}\chi_A(z)\,dm). \qquad (2)$$

For k sufficiently large, let $z_k \in A$, such that: $\sigma^k(z_k) = y_0$, and

$$R_-^k(z_k) > \sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z) - \frac{1}{2k}.$$

We denote $z_k := x_k...x_1y_0$, $x_i \in M$. We can take $\epsilon > 0$ sufficiently small such that the ball $A_{k,\epsilon} := \{a_k...a_1 \in m^k : |(a_k...a_1) - (x_k...x_1)| \leq \epsilon\}$ satisfies:
1. $a_k...a_1y_0 \in A$,
2. $a_k...a_1y_{c,k} \in A$, for $k, c >> 0$
3. $R_-^k(a_k...a_1y_0) > \sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z) - \frac{1}{k}$

Then we have:

$$\limsup_{k\to\infty}\liminf_{c\to\infty} \frac{1}{c}\log(\int_{A_{k,\epsilon}} e^{cR_-^k(a_k...a_1y_0)}\,dm)$$

$$\geq \limsup_{k\to\infty} \liminf_{c\to\infty} \Big(\sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z) - \frac{1}{k} \Big) + \frac{1}{c}\log(m(A_{k,\epsilon}))$$

$$= \limsup_{k\to\infty} \Big(\sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z) \Big) + \limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c}\log(\epsilon)$$

$$= \limsup_{k\to\infty} \Big(\sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z) \Big). \tag{3}$$

By other hand, on $A_{k,\epsilon}$ we have:

$$R_-^k(a_k...a_1 y_{c,k}) \geq R_-^k(a_k...a_1 y_0) - |R_-|_\theta(\theta+\theta^2+...+\theta^k) d(y_{c,k}, y_0)$$

$$\geq R_-^k(a_k...a_1 y_0) - \frac{|R_-|_\theta d(y_{c,k}, y_0)}{1-\theta}.$$

Then, we get

$$\limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c}\log\Big(\int_{\sigma^k(z)=y_{c,k}} e^{cR_-^k(z)} \chi_A(z)\, dm \Big)$$

$$\geq \limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c}\log\Big(\int_{A_{k,\epsilon}} e^{cR_-^k(a_k...a_1 y_{c,k})}\, dm \Big)$$

$$\geq \limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c}\log\Big(\int_{A_{k,\epsilon}} e^{cR_-^k(a_k...a_1 y_0) - c\frac{|R_-|_\theta d(y_{c,k},y_0)}{1-\theta}}\, dm \Big)$$

$$= \limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c}\log\Big(\int_{A_{k,\epsilon}} e^{cR_-^k(a_k...a_1 y_0)}\, dm \Big). \tag{4}$$

Using (2), (4) and (3), we finish the proof. □

Now we fix the point y_0 given above. The next result is basically contained in the proof of proposition 5 in [3].

Lemma 13. *Let p be a point on the support of μ_∞. Let y_n a sequence satisfying $\sigma(y_n) = y_{n-1}, n = 1, 2, 3, ...,$ and, $0 = R_-(y_1) = R_-(y_2) = ...$ (it follows from the property of the calibrated subaction). Then p is a accumulation point of $\{y_n\}$.*

Proof. Let B be the set of accumulation points of $\{y_n\}$. B is closed and $\sigma(B) = B$. Then there exists a invariant probability ν with support on B. The inclusion $B \subseteq X$ implies the existence of an extension of ν to X by the rule: $\nu(\phi) := \nu(\phi.\chi_B)$. Using the fact that R_- is a continuous function, and, that $R_-(y_n) = 0, n = 1, 2, ...,$ we conclude that $\chi_B.R_- = 0$. So, $\nu(R_-) = 0$, and then, $\nu = \mu_\infty$. From this we get that the support of μ_∞ is contained on B. □

The next lemma follows the same reasoning of Lemma 18 in [22]:

Lemma 14. *If $R_-^\infty(z) > -\infty$, then the family of probabilities ν_n (also called empirical measures), given by $\phi \to \frac{1}{n}\sum_{j=0}^{n-1}\phi(\sigma^j(z))$, converges to μ_∞ weakly*, when $n \to \infty$.*

Proof. Any accumulation measure of ν_n is an invariant probability. We are going to show that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\sigma^j(z)) \geq \mu_\infty(f).$$

Let $M = R_-^\infty(z)$. Then, for each n we have $R_-^n(z) \geq M$, so:

$$V(z) - V(\sigma^n(z)) - n\mu_\infty(f) + \sum_{j=0}^{n-1} f(\sigma^j(z)) = \sum_{j=0}^{n-1} R_-(\sigma^j(z)) = R_-^n(z) \geq M.$$

Then, we get

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\sigma^j(z)) \geq \frac{M}{n} - \frac{2|V|_\infty}{n} + \mu_\infty(f).$$

Finally, taking $\liminf\limits_{n\to\infty}$ in the above we show the claim. $\square$

Corollary 15. *If $R_-^\infty(z) > -\infty$, and, $p \in supp(\mu_\infty)$, then p is an accumulation point of $\sigma^n(z)$.*

Proof. Let $p \in supp(\mu_\infty)$, and $\varepsilon > 0$. Consider the ball $B(p,\varepsilon) := \{x \in X : d(x,p) < \varepsilon\}$. Using the fact that $p \in supp(\mu_\infty)$, we have that $\mu_\infty(B(p,\varepsilon)) > 0$. So, by the above lemma, we have that $\{\sigma^n(z)\}$ is in the ball for infinite values of n. $\square$

Lemma 16.

$$\sup_{z\in A} R_-^\infty(z) \leq \limsup_{k\to\infty}\big(\sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z)\big).$$

Proof. We fix a point $p \in supp(\mu_\infty)$, and, we denote $p = p_1p_2....$ For $n \geq n_0$, there exists $y \in A$, with $\sigma^n(y) = p$. Note that $R_-^\infty(y) = R_-^n(y) > -\infty$. So, $\sup_{z\in A} R_-^\infty(z) > -\infty$. Let $z_0 \in A$ be such that, $R_-^\infty(z_0) > -\infty$, and, denote $z_0 = x_1x_2....$ Given $t \in \mathbb{N}$, let $n(t)$ be such that, $d(\sigma^{n(t)}(z_0), p) \leq \theta^t$, and moreover, such that, the choice $x_1x_2...x_{n(t)}$, determines that $z_0 \in A$ (open). By lemma 13 there exist a pre-image of y_0 (we suppose the point $y_{l(t)}$ of the form $a_{l(t)}...a_1(y_0)$), such that, $R_-^{l(t)}(y_{l(t)}) = 0$, and, $d(y_{l(t)}, p) < \theta^t$. Define $z(t)$ by

$$z(t) := x_1...x_{n(t)}a_{l(t)}...a_1(y_0).$$

Then, we have:

$$\sup_{z\in A:\sigma^{l(t)+n(t)}(z)=y_0} R_-^{l(t)+n(t)}(z) \geq R_-^{l(t)+n(t)}(z(t))$$
$$= R_-^{n(t)}(z(t)) + R_-^{l(t)}(y_{l(t)}) = R_-^{n(t)}(z(t))$$
$$\geq R_-^{n(t)}(z_0) - |R_-|_\theta 2(\theta^t + \theta^{t+1} + ... + \theta^{t+n(t)})$$
$$\geq R_-^{n(t)}(z_0) - 2\frac{\theta^t|R_-|_\theta}{1-\theta} \geq R_-^{\infty}(z_0) - 2\frac{\theta^t|R_-|_\theta}{1-\theta}.$$

So, when $t \to \infty$

$$\limsup_{k\to\infty}(\sup_{z\in A,\,\sigma^k(z)=y_0} R_-^k(z)) \geq \limsup_{t\to\infty} \sup_{z\in A:\sigma^{l(t)+n(t)}(z)=y_0} R_-^{l(t)+n(t)}(z)$$
$$\geq R_-^{\infty}(z_0).$$

Using that z_0 is arbitrary, and satisfies $R_-^{\infty}(z_0) > -\infty$, we conclude the proof. □

From this result and lemma 12 we get:

Proposition 17.

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^n\chi_A)(x)) \geq \sup_{z\in A} R_-^{\infty}(z).$$

This conclude the proof of Theorem 8.

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