Selection of measure and a Large Deviation Principle for the general one-dimensional XY model

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Abstract

We consider (M,d) a connected and compact manifold and we denote by X the Bernoulli space $M^{\mathbb{N}}$. The shift acting on X is denoted by σ.

We analyze the general XY model, as presented in a recent paper by A. T. Baraviera, L. M. Cioletti, A. O. Lopes, J. Mohr and R. R. Souza. Denote the Gibbs measure by $\mu_c := h_c \nu_c$, where h_c is the eigenfunction, and, ν_c is the eigenmeasure of the Ruelle operator associated to cf. We will show that any measure selected by μ_c, as $c \to +\infty$, is a maximizing measure for f. We also prove, when the maximizing probability measure is unique, that it is true a Large Deviation Principle with the deviation function $R_+^\infty = \sum_{j=0}^{\infty} R_+(\sigma^j)$, where $R_+ := \beta(f) + V \circ \sigma - V - f$ and V is any calibrated subaction.

1 Introduction

We consider (M,d) a connected and compact finite dimensional manifold and we denote by X the Bernoulli space $M^{\mathbb{N}}$. The shift acting on X is denoted by σ.

We point out that the number of preimages by σ of each point is not countable.

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Let $f : X \to \mathbb{R}$ be a fixed Holder potential defined in the Bernoulli space X. We denote by m the Lebesgue probability on M. We suppose without loss of generality that the diameter of the manifold M is smaller than one. This distance induces another one, in the usual fashion, on $M^{\mathbb{N}}$ [7].

We are interested in the Gibbs state (for finite and zero temperature) associated to the potential f. This model is called the general XY model in [7]. We refer the reader to such work for a detailed explanation about the motivation for considering such kind of problems. We point out that in the literature in Physics what is called the XY model is the case when $M = S^1$, and, the potential f depends just on a finite number of coordinates. In [7] and here the hypothesis are more general.

Classical references in the XY model are [20], [34] and [38] where the spin can be in a circle. A nice reference for general results in Statistical Mechanics is [18].

In order to define a transfer operator we need a probability a priori on M which we will denote by dm. In the case $M = S^1$ is usual to consider the Lebesgue measure dx (see [38]) as the a priori probability. In [31] it is consider the case of the one dimensional spin lattice with a general a priori probability and is presented results which are generalizations of the ones in [7].

First we will recall some definitions and results from [7].

Definition 1. *Let $\mathcal{C}$ be the space of continuous functions from $X = M^{\mathbb{N}}$ to $\mathbb{R}$. We define the Ruelle operator L_f acting on $\mathcal{C}$, associated to the Holder potential $f : M^{\mathbb{N}} \to \mathbb{R}$, as the linear operator that takes $w \in \mathcal{C}$, and send it to $L_f(w) \in \mathcal{C}$, which is defined for any $x = (x_0, x_1, x_2, ...) \in X$, in the following way*

$$L_f(w)(x) = \int e^{f(ax)}\, w(ax)\, dm\,(a)\,,$$

where ax represents the sequence $(a, x_0, x_1, x_2, ...) \in X$, and $dm\,(a)$ is the Lebesgue probability on M.

Following [7], for a real value c we consider β_c the main eigenvalue, h_c the associated eigenfunction, and $g_c = cf + \log(h_c) - \log(h_c \circ \sigma) - \log(\beta_c)$ the normalized function associated to the Ruelle operator L_{cf} obtained from cf. We also denote by ν_c the eigenmeasure of L^*_{cf}, and, by $\mu_c := h_c \nu_c$, the Gibbs probability of the potential cf.

Note that if for each point there exists an uncountable number of preimages, then it is necessary to use an a priori probability m in Definition 1.

As usual, by notation $f^n(x) = \sum_{j=0}^{n-1} f(\sigma^j(x))$, for any $n \in \mathbb{N}$, $x \in X$.

Remark on notation: the iterated Ruelle Operator $L_f^n w(x)$, $n = 1, 2, 3...$, can be written as

$$\int_{a_n...a_1} e^{f^n(a_n...a_1 x)} w(a_n...a_1 x)\, da_1...da_n, \quad or \quad \int_{\sigma^n z = x} e^{f^n(z)} w(z)\, dm.$$

The Bernoulli space $M^{\mathbb{N}}$ has the structure of a differentiable infinite dimensional manifold. In this new setting of Thermodynamic Formalism one can ask different kinds of questions: if the potential is differentiable, is it true that there exists a positive differentiable main eigenfunction. This question was addressed in [31]. Our focus here is in a different class of problems.

We denote by $\mathcal{M}_\sigma$ the compact set of invariant probability measures for σ.

We consider the following problem: for a given $f : X \to \mathbb{R}$, we want to find probability measures which maximize, over $\mathcal{M}_\sigma$, the value $\int f(x)\, d\mu(\mathbf{x})$.

Definition 2. *We define*

$$\beta(f) = \max_{\mu \in \mathcal{M}_\sigma} \left\{ \int f d\mu \right\}.$$

Any of the measures which attains the maximal value will be called a maximizing probability measure for f, which is sometimes denoted by μ_∞.

There exist Holder potentials f such that the maximizing probability is not unique. In case of the shift acting on the Bernoulli space $\{1, 2, ., d\}^{\mathbb{N}}$, it is known that for a generic potential f in the Holder norm the maximizing probability is unique (see [14]). It can be for instance a probability with support in a periodic orbit. A basic reference for the classical study of maximizing probabilities is [3].

It is known that for any fixed $x \in M^{\mathbb{N}}$ the probabilities $\mu_{n,c}$ defined by

$$\int w\, d\mu_{n,c} := L_{g_c}^n(w)(x)$$

converges to μ_c in the weak* topology, when, $n \to \infty$ (see for instance [7]).

The Classical Thermodynamic Formalism setting considers the shift acting on the Bernoulli space $\{1, 2, ., d\}^{\mathbb{N}}$ (see [37]). In this case one can consider the classical Kolmogorov entropy and pressure. In this way, the variational principle can be used to prove that any limit of the the Gibbs measure μ_c (when $c \to \infty$) is a maximizing measure to f (see Prop. 29 in [14] or [13]). In order to prove this last result in the present setting we use a different approach: we consider the double limit[1] for $\mu_{n,c}$.

In Section 2 we will show the following:

[1]Given a double indexed sequence $z_{c,n}$, $c \in \mathbb{R}$, $n \in \mathbb{N}$, we say that $\lim_{c,n\to\infty} z_{c,n} = w$, if for any given $\epsilon > 0$, there exists an $M > 0$, such that, if $c, n > M$, then $|z_{c,n} - w| < \epsilon$.

Theorem 3. *Any weak*-limit of subsequence of* μ_c*, $(c \to +\infty)$, is a maximizing probability measure for* f*. If the maximizing probability* μ_∞ *for* f *is unique, then* $\mu_c \to \mu_\infty$*, when* $c \to \infty$*.*

It is known that the parameter c corresponds to the inverse of temperature in Statistical Mechanics. In this way one can say that any convergent subsequence of Gibbs states at positive temperature, when $c \to \infty$, selects at zero temperature maximizing probabilities. For this result we do not assume uniqueness of the maximizing probability.

A ground state is by definition a maximizing probability that can be "selected" as the limit of μ_{c_n} for some subsequence $c_n \to \infty$. There are examples of Holder potentials f where certain maximizing probabilities are not ground states (see [10], [24], [4], [5]). From the above we realize that maximizing probabilities are related to the concept of "Gibbs state at temperature zero". A maximizing probability with support in a periodic orbit corresponds to magnetization. Gibbs states at positive temperature for Holder potentials does not have such property.

In the present case a definition of entropy is possible and this is described in detail in [31]. In this case the entropy is a non positive number (see also [32]). The main conclusions one can get from [31] is that: I) In the concept of Kolmogorov entropy there exist an a priori measure which is hidden; II) Entropy and Ruelle operator are concepts which are linked.

Here we will not take advantage of the approach described in [31] and our proof will use other more simple methods, where we consider the double limit for $L^n_{gc}(\cdot)(x)$.

Definition 4. *A continuous function* $V : X \to \mathbb{R}$ *is called a* calibrated subaction *for* $f : X \to \mathbb{R}$*, if for any* $y \in X$*, we have that*

$$V(y) = \max_{\sigma(x)=y} [f(x) + V(x) - \beta(f)]. \tag{1}$$

This can be also be expressed as

$$\beta(f) = \max_{a \in M} \{f(ay) + V(ay) - V(y)\}.$$

One can show that for any x in the support of a maximizing probability measure for f we have that

$$V(\sigma(x)) - V(x) - f(x) + \beta(f) = 0.$$

In this way if we know the value $\beta(A)$, then a calibrated subaction V for f helps to identify the union of the supports of maximizing probabilities μ_∞

for f. The above equation can be eventually true outside the union of the supports of the maximizing probabilities μ.

It is known that $\frac{1}{c}\log(h_c)$, $c \in \mathbb{R}$, is a equicontinuous family. Any limit of subsequence $V = \lim_{n\to\infty} \frac{1}{c_n}\log(h_{c_n})$, $c_n \to \infty$, is a calibrated subaction (see [7]). If the maximizing probability is unique, then the calibrated subaction is unique up to an additive constant (see [6] [17]). In this way we write $R_+^\infty = \sum_{j=0}^\infty R_+(\sigma^j)$, with $R_+ := \beta(f) + V\circ\sigma - V - f$, where V is any chosen calibrated subaction.

In section 3 we consider the case where the maximizing measure for f is unique, and prove that the family $L^n_{g_c}(\cdot)(x)$ satisfies the following kind of Large Deviation Principle:

Theorem 5. *Suppose that the maximizing probability for f is unique. Consider any point $x \in X$, then, for any closed set F, and any open set A:*

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \leq -\inf_{z\in F}(R_+^\infty(z)),$$

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L^n_{g_c}\chi_A)(x)) \geq -\inf_{z\in A}(R_+^\infty(z)).$$

The function $I = R_+^\infty$ is lower semicontinuous, non negative and zero at least in the support of the maximizing probability.

The estimation of the asymptotic above via the Ruellle operator is useful because the computations of most of the important objets in Thermodynamic Formalism are obtained via this operator.

From the last theorem, taking x fixed, $n \to \infty$, and using the fact that $L^n_{g_c}(\cdot)(x)$ converges to μ_c weakly* (see [7]) we get the following:

Corollary 6. *Suppose $M = \mathbb{S}^1$. For any cylinder $F = I_1 \times ... \times I_n \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times ....$, where each I_j is a closed interval of $\mathbb{S}^1$ we have*

$$\limsup_{c\to\infty} \frac{1}{c}\log(\mu_c(F)) \leq -\inf_{z\in F}(R_+^\infty(z)).$$

For any cylinder $A = I_1 \times ... \times I_n \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times ....$, where each I_j is a open interval of $\mathbb{S}^1$ we have

$$\liminf_{c\to\infty} \frac{1}{c}\log(\mu_c(A)) \geq -\inf_{z\in A}(R_+^\infty(z)).$$

We point out that the reasonings which we use in the proofs of the above results can also be applied to the Classical Thermodynamic Formalism setting [37], where the Bernoulli space is $\{1, 2, ..., d\}^{\mathbb{N}}$, to get the analogous result presented in [6]. The present proof of the L. D. P. does not use the involution kernel as in [6].

We say that the potential A depends on two variables if for any $x = (x_0, x_1, x_2, x_3....) \in X$ we have that the value $A(x_0, x_1, x_2, x_3....)$ is independent of $(x_2, x_3, x_4, ...)$. This case is also known as the "nearest-neighbour interaction". The last corollary was shown to be true in the case the potential A depends on two coordinates in [30].

There are two kinds of Large Deviations which are more commonly study:

I) Large deviations in time - In this case one is interested in results like: given an ergodic invariant measure μ for the shift (or another dynamical system) acting on $\{1, 2, ., d\}^{\mathbb{N}}$, then, what is the probability on x that a given Birkhoff sum $\frac{1}{n}\sum_{j=0}^{n-1}\varphi(\sigma^j(x))$ deviates of the mean value $\int \varphi\, d\mu$ by, let's say, an error of ϵ? This is what [16] call Level 1 L. D. P..

General references for this kind of dynamical results are for instance [41], [23], [26] and [27]. These kind of problems can be studied in a more broad sense for general stochastic processes (see [16], [15]). The main point in the "dynamical setting of expanding maps and Gibbs probabilities for Holder potentials" is that one can exhibit in an explicit form the deviation function (via Legendre transform of the pressure) which is analytic. Differentiability of pressure plays an important role in all this.

In [39] the authors show that the Ruelle operator is an analytic function of the Holder potential and this result can be applied to the present situation. Therefore, we believe it is just a question of rewriting the classical arguments of Thermodynamic Formalism on our setting in order to show that indeed is true a Large Deviation principle in time for the XY model.

II) Large deviations for measures indexed by a parameter $c \in \mathbb{R}$ - General references for this kind of problem are [40](see section 0. Introduction) and [15]. In this last book in section 4.3 the parameter $\epsilon \to 0$ corresponds here to $c = 1/\epsilon \to \infty$.

Consider a family of probabilities μ_ϵ such that, $\lim_{\epsilon\to 0}\mu_\epsilon = \mu$. Given a set J such that $\mu(J) = 0$ one can ask about the exponential velocity such that $\mu_\epsilon(J)$ goes to zero, as $\epsilon \to 0$. This is the other kind of Large Deviation which is also very much analyzed.

Returning to the dynamical L. D. P of case I), but now considering the empirical probability $\mu_n(x) = \frac{1}{n}\sum_{j=0}^{n-1}\delta_{\sigma^j(x)}$, it is known that for μ almost every x we have $\mu_n(x) \to \mu$. One can ask about deviations of this empirical probability from a small open set containing μ when $n \to \infty$. This is what [16] call Level 2 L. D. P.. This question is analyzed in [26] and [23] for the case μ is the measure of maximal entropy. In this case there exists a deviation function I which is defined in the set of probabilities ν and which is given by

$I(\nu) = \log d - h(\nu) \geq 0$, for invariant ones, and $-\infty$ in the other case. This is one more evidence that it would be more natural to consider Kolomogorov entropy with respect to an a priori probability and not with respect to the counting measure on $\{1, 2, ., d\}$. Entropy is by its very nature a non negative number, namely $h(\nu) - \log d \leq 0$. Well, one can say that to substract or not a constant is just a question of convention. However, note that this can not be done if dynamical system under consideration is such that each point has an infinite number of preimages (see [31]). Anyway, this setting looks more like the case of L. D. P. we consider in the present paper. The deviation function is quite irregular.

The famous theorem of Schilder concerns a renormalization of time of the Probability of the Brownian motion by a small parameter ϵ (see Theorem 5.2.3 in [15]). In Aubry-Mather theory this result is applied to the study of viscosity solutions, subactions and Large Deviations associated to diffusions (see Appendix in [2] and [1]).

We point out that here (in a similar way as in [6]) we are able to express the deviation function I in an explicit form from the information given by the calibrated subaction. This deviation function I is quite irregular: it is just lower semicontinuous and equal to $-\infty$ in most of the points. The bottom line is: L. D. P in II) is quite different from I).

Most of the ideas of the present paper are generalizations of results which are contained in [35] where was considered just the classical case of the shift acting on $\{1, 2, .., d\}^{\mathbb{N}}$.

In [29] is presented another kind of Large Deviation Principle: the setting of zeta measures. In this case the proof does not require that the maximizing probability is unique.

2 The selection of measure

Lemma 7. *Let V be a calibrated subaction, such that, $V = \lim_{c\to\infty} \frac{1}{c}\log(h_c)$, and, $R_- = f + V - V \circ \sigma - \beta(f)$, which is the limit function of the g_c/c associated. For each $\epsilon > 0$, there exists a constant ψ_ϵ, such that, for any $x \in X$*

$$m(\{a \in M : R_-(ax) > -\epsilon\}) > \psi_\epsilon > 0.$$

Proof. Suppose g_c converges to R_-, and then write $g_c = cR_- + \delta_c$, where $|\delta_c|_\infty/c \to 0$. Using the fact that V is a calibrated subaction, we have that $R_- \leq 0$.

We fix $\epsilon > 0$, and we define

$$A_\epsilon := \{a : R_-(ax) \leq -\epsilon\}$$

$$B_\epsilon := \{a : R_-(ax) > -\epsilon\}.$$

V is Holder, so R_- is Holder, then, it is a continuous function on the first symbol. In this way, A_ϵ and B_ϵ are mensurable sets. Then, we get

$$1 = L_{g_c} 1(x) = \int e^{g_c(ax)} da = \int e^{cR_-(ax)+\delta_c(ax)} da.$$

Therefore,

$$\begin{aligned} 1 &= \int_{A_\epsilon} e^{cR_-(ax)+\delta_c(ax)} da + \int_{B_\epsilon} e^{cR_-(ax)+\delta_c(ax)} da \\ &\leq \int_{A_\epsilon} e^{-c\epsilon+\delta_c(ax)} da + \int_{B_\epsilon} e^{0+\delta_c(ax)} da \\ &\leq \int_{A_\epsilon} e^{-c\epsilon+|\delta_c|_\infty} da + \int_{B_\epsilon} e^{0+|\delta_c|_\infty} da \\ &= e^{-c\epsilon+|\delta_c|_\infty} m(A_\epsilon) + e^{|\delta_c|_\infty} m(B_\epsilon) \\ &\leq e^{-c\epsilon+|\delta_c|_\infty} + e^{|\delta_c|_\infty} m(B_\epsilon). \end{aligned}$$

Let $c_0 > 0$ be such that $e^{-c_0\epsilon+|\delta_{c_0}|_\infty} \leq 1/2$. Then, it follows that

$$1/2 \leq e^{|\delta_{c_0}|_\infty} m(B_\epsilon),$$

so,

$$m(B_\epsilon) \geq \frac{1}{2e^{|\delta_{c_0}|_\infty}}.$$

Then, we just take $\psi_\epsilon = \frac{1}{3e^{|\delta_{c_0}|_\infty}}$, and the result follows. □

Proof of Theorem 3

Proof. Let ν be an accumulation point of μ_c given by the limit of a certain subsequence $c_j \to \infty$. Let c_i be a subsequence obtained from this first one such that there exists the limit V of the sequence $\frac{1}{c_i} \log h_{c_i}$. Let $R_- := f + V - V \circ \sigma - \beta(f)$ be the function associated to such limit. Then, we get $g_{c_i}/c_i \to R_-$. Define $a := \lim_{c_i\to\infty} \mu_{c_i}(R_-) = \lim_{c_j\to\infty} \mu_{c_j}(R_-)$. Then, it follows that $a \leq 0$. We will show that $a \geq 0$. More precisely we are going to show that for any fixed $x \in X$:

$$\liminf_{i,n\to\infty} L^n_{g_{c_i}}(R_-)(x) \geq 0.$$

We write $\log h_{c_i} = c_i V + \delta_i$, where

$$\frac{|\delta_i|_\infty}{c_i} \to 0.$$

We also write

$$L^n_{g_{c_i}}(R_-)(x) = \int_{a_n...a_1} e^{g^n_{c_i}(a_n...a_1x)} R_-(a_n....a_1x)da_1...d_{a_n}$$

$$= \frac{\int_{a_n...a_1} e^{c_if^n(a_n...a_1x)+\log(h_{c_i}(a_n...a_1x))-\log(h_{c_i}(x))} R_-(a_n...a_1x)da_1...da_n}{\int_{a_n...a_1} e^{c_if^n(a_n...a_1x)+\log(h_{c_i}(a_n...a_1x))-\log(h_{c_i}(x))} da_1...da_n}$$

$$= \frac{\int_{a_n...a_1} e^{c_if^n(a_n...a_1x)+\log(h_{c_i}(a_n...a_1x))} R_-(a_n...a_1x)da_1...da_n}{\int_{a_n...a_1} e^{c_if^n(a_n...a_1x)+\log(h_{c_i}(a_n...a_1x))} da_1...da_n}$$

$$= \frac{\int_{a_n...a_1} e^{c_if^n(a_n...a_1x)+c_iV(a_n...a_1x)+\delta_i(a_n...a_1x)-c_iV(x)-c_i\beta(f)} R_-(a_n...a_1x)da_1...da_n}{\int_{a_n...a_1} e^{c_if^n(a_n...a_1x)+c_iV(a_n...a_1x)+\delta_i(a_n...a_1x)-c_iV(x)-c_i\beta(f)} da_1...da_n}$$

$$= \frac{\int_{a_n...a_1} e^{c_iR^n_-(a_n...a_1x)+\delta_i(a_n...a_1x)} R_-(a_n...a_1x)da_1...da_n}{\int_{a_n...a_1x} e^{c_iR^n_-(a_n...a_1x)+\delta_i(a_n...a_1x)} da_1...da_n}.$$

For a fixed $\varepsilon > 0$, we define the sets:

$$A_n := \{a_n...a_1 : R_-(a_n...a_1x) < -\varepsilon\},$$
$$B_n := \{a_n...a_1 : R_-(a_n...a_1x) \geq -\varepsilon\},$$
$$C_n := \{a_n...a_1 : R_-(a_n...a_1x) > -\frac{\epsilon}{2}\}.$$

Clearly, we have that $C_n \subseteq B_n$.

As V is a calibrated subaction, then C_n is not empty. We remark that $A_n \cup B_n = M^n$, and, by the Lemma above, for each $a_{n-1}...a_1$, we have that $m\{a_n : a_n...a_1 \in C_n\} > \psi_{\epsilon/2} > 0$.

Then, we get

$$\int_{A_n} e^{c_iR^n_- + \delta_i} da_1...da_n = \int_{A_n} e^{c_iR_- + \delta_i} e^{c_iR^{n-1}_- \circ\sigma} da_1...da_n$$

$$\leq \int_{A_n} e^{-c_i\varepsilon + |\delta_i|_\infty} e^{c_iR^{n-1}_-\circ\sigma} da_1...da_n$$

$$= e^{-c_i\varepsilon+|\delta_i|_\infty} \int_{A_n} e^{c_iR^{n-1}_-\circ\sigma} da_1...da_n$$

$$\leq e^{-c_i\varepsilon+|\delta_i|_\infty} \int_{M^n} e^{c_iR^{n-1}_-\circ\sigma} da_1...da_n$$

$$= e^{-c_i\varepsilon+|\delta_i|_\infty} \int_{M^{n-1}} e^{c_iR^{n-1}_-} da_1...da_{n-1},$$

and,

$$
\begin{aligned}
\int_{B_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n &= \int_{B_n} e^{c_i R_- + \delta_i} e^{c_i R_-^{n-1} \circ \sigma} da_1 ... da_n \\
&\geq \int_{C_n} e^{c_i R_- + \delta_i} e^{c_i R_-^{n-1} \circ \sigma} da_1 ... da_n \\
&\geq \int_{C_n} e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} e^{c_i R_-^{n-1} \circ \sigma} da_1 ... da_n \\
&\geq e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \int_{C_n} e^{c_i R_-^{n-1} \circ \sigma} da_1 ... da_n \\
&\geq e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \int_{M^{n-1}} \int_{\{a_n : a_n ... a_1 \in C_n\}} e^{c_i R_-^{n-1} \circ \sigma} da_1 ... da_n \\
&= e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \int_{M^{n-1}} e^{c_i R_-^{n-1}} \int_{\{a_n : a_n ... a_1 \in C_n\}} da_1 ... da_n \\
&= e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty} \psi_{\epsilon/2} \int_{M^{n-1}} e^{c_i R_-^{n-1}} da_1 ... da_{n-1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 &\leq \liminf_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n} \\
&\leq \limsup_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n} \\
&\leq \limsup_{i,n\to\infty} \frac{e^{-c_i \varepsilon + |\delta_i|_\infty}}{\psi_{\epsilon/2} e^{-c_i \frac{\epsilon}{2} - |\delta_i|_\infty}} \\
&\leq \limsup_{i,n\to\infty} e^{-c_i \varepsilon/2 + 2|\delta_i|_\infty} \psi_{\epsilon/2}^{-1} \\
&= \limsup_{i,n\to\infty} e^{c_i (-\epsilon/2 + 2\frac{|\delta_i|_\infty}{c_i})} \psi_{\epsilon/2}^{-1} = 0.
\end{aligned}
$$

In the same way

$$
\begin{aligned}
0 &\leq \liminf_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} R_- \; da_1 ... da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n (-\varepsilon)} \\
&\leq \limsup_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n + \delta_i} R_- \; da_1 ... da_n}{\int_{B_n} e^{c_i R_-^n + \delta_i} da_1 ... da_n (-\varepsilon)}
\end{aligned}
$$

$$\leq \limsup_{i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n+\delta_i} da_1...da_n\,(-|R_-|_\infty)}{\int_{B_n} e^{c_i R_-^n+\delta_i} da_1...da_n(-\varepsilon)}$$

$$\leq \limsup_{i,n\to\infty} \frac{e^{-c_i\varepsilon+|\delta_i|_\infty}(-|R_-|_\infty)}{\psi_{\epsilon/2}(-\varepsilon)e^{-c_i\,\epsilon/2\,-\,|\delta_i|_\infty}} = 0.$$

From the above, and writing $\int_{M^n} da_1...da_n = \int_{A_n} da_1...da_n + \int_{B_n} da_1...da_n$, we have:

$$\liminf_{c_i,n\to\infty} L^n_{g_{c_i}}(R_-)(x) = \liminf_{c_i,n\to\infty} \frac{\int_{a_n...a_1} e^{c_i R_-^n(a_n...a_1x)+\delta_i(a_n...a_1x)} R_-(a_n...a_1x)da_1...da_n}{\int_{a_n...a_1} e^{c_i R_-^n(a_n...a_1x)+\delta_i(a_n...a_1x)} da_1...da_n}$$

$$\geq \liminf_{c_i,n\to\infty} \frac{\int_{A_n} e^{c_i R_-^n+\delta_i} R_- da_1...da_n + \int_{B_n} e^{c_i R_-^n+\delta_i} da_1...da_n(-\varepsilon)}{\int_{A_n} e^{c_i R_-^n+\delta_i} da_1...da_n + \int_{B_n} e^{c_i R_-^n+\delta_i} da_1...da_n}$$

$$= \liminf_{c_i,n\to\infty} \frac{\int_{B_n} e^{c_i R_-^n+\delta_i} da_1...da_n(-\varepsilon)}{\int_{B_n} e^{c_i R_-^n+\delta_i} da_1...da_n}$$

$$\geq -\varepsilon.$$

Taking $\varepsilon \to 0$, we get our claim. □

This ends the proof of our first main result. The bottom line is: a convergent subsequence of Gibbs states at positive temperature selects maximizing probabilities (eventually, different subsequences can localize different probabilities).

3 On the Large Deviation Principle

On this section we are going to prove Theorem 5.

We suppose that the maximizing measure for f is unique, and we denote by μ_∞ the maximal one. Under this assumption, two calibrated subactions differ by a constant (this follows from proposition 5 in [6]). In particular the function $R_+ := \beta(f)+V\circ\sigma-V-f$ is well defined. The function $R_- := -R_+$ is the unique accumulation point of g_c/c, on the uniform topology, so $g_c/c \to R_-$ uniformly.

We denote by $|g|_\theta$ the Lipschitz constant of a Lipschitz function g.

Lemma 8. *The function R_+^∞ is lower semi-continuous.*

Proof. We take $z, z_j \in X$, with $z_j \to z$. We will show that

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq R_+^\infty(z).$$

In the case $R_+^\infty(z) = 0$ the result is clearly true.

First case: $R_+^\infty(z) = \infty$.
Given $M > 0$, let n be such that $R_+^n(z) > 2M$. We fix n_1, such that, $\frac{M}{2^{n_1}|R_+|_\theta} < 1$. Let n_0 be such that for $j \geq n_0$, we have $d(z_j, z) < \frac{M}{2^{n_1}2^n|R_+|_\theta}$. Then, for $j \geq n_0$:

$$\begin{aligned} R_+^n(z_j) &\geq R_+^n(z) - |R_+|_\theta(d(z_j, z) + ... + d(\sigma^{n-1}(z_j), \sigma^{n-1}z)) \\ &\geq 2M - |R_+|_\theta(\frac{M}{2^{n_1}|R_+|_\theta}) \geq M. \end{aligned}$$

It follows that

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq M.$$

Taking $M \to +\infty$, we get

$$\liminf_{j\to\infty} R_+^\infty(z_j) = +\infty.$$

Second case: $R_+^\infty(z) = M > 0$.
Fixed $\varepsilon > 0$, there exist n, such that, $R_+^n(z) > M - \varepsilon/2$. Let n_0 be such that for $j \geq n_0$, then we have $d(z_j, z) < \frac{\varepsilon}{2^{n+2}|R_+|_\theta}$. Therefore, for $j \geq n_0$:

$$R_+^n(z_j) \geq (M - \varepsilon/2) - |R_+|_\theta \left(\frac{\varepsilon}{2|R_+|_\theta}\right) = M - \varepsilon.$$

Therefore, we get

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq M - \varepsilon.$$

Taking $\varepsilon \to 0$:

$$\liminf_{j\to\infty} R_+^\infty(z_j) \geq M.$$

□

We note that in [7] it is proved that $\frac{1}{c}\log(\beta_c) \to \beta(f)$. We denote $\varepsilon_c = \log(\beta_c) - c\beta(f)$. Then, we have $\frac{\varepsilon_c}{c} \to 0$.

Lemma 9.

$$\lim_{c,n\to\infty}\left(\frac{1}{c}\log((L^n_{cR_-}1)(x))-\frac{n.\varepsilon_c}{c}\right)=0,$$

in particular, for a fixed k:

$$\lim_{c,n\to\infty}\frac{1}{c}\log((L^n_{cR_-}1)(x))-\frac{1}{c}\log((L^{n+k}_{cR_-}1)(x))=0.$$

Proof. Let a be an accumulation point of $\frac{1}{c}\log((L^n_{cR_-}1)(x))-\frac{n.\varepsilon_c}{c}$, when $c,n\to\infty$. Then, there exists $c_j,n_j\to\infty$, such that,

$$\lim_{j\to\infty}\left(\frac{1}{c_j}\log((L^{n_j}_{c_jR_-}1)(x))-\frac{n_j.\varepsilon_{c_j}}{c_j}\right)=a.$$

Following [7] we can take a subsequence $\{j_i\}$ such that $\dfrac{1}{c_{j_i}}\log(h_{c_{j_i}})$ converges uniformly to a calibrated subaction V. So there exist sequences $c_i,n_i\to\infty$ such that:

$$\lim_{i\to\infty}\left(\frac{1}{c_i}\log((L^{n_i}_{c_iR_-}1)(x))-\frac{n_i.\varepsilon_{c_i}}{c_i}\right)=a,\ \text{ and },\ \lim_{i\to\infty}\frac{1}{c_i}\log(h_{c_i})=V.$$

Denoting $\log(h_{c_i})=c_iV+\delta_{c_i}$ where $|\delta_{c_i}|_\infty/c_i\to 0$, we have:

$$\begin{aligned}
0&=\lim_{i\to\infty}\frac{1}{c_i}\log((L^{n_i}_{g_{c_i}}1)(x))\\
&=\lim_{i\to\infty}\frac{1}{c_i}\log(\int_{\sigma^{n_i}(z)=x}e^{c_if^{n_i}(z)+\log(h_{c_i}(z))-\log(h_{c_i}(x))-n_i\log(\beta_{c_i})}\,dm)\\
&=\lim_{i\to\infty}\frac{1}{c_i}\log(\int_{\sigma^{n_i}(z)=x}e^{c_if^{n_i}(z)+c_iV(z)-c_iV(x)-n_ic_i\beta(f)+\delta_{c_i}(z)-\delta_{c_i}(x)-n_i\varepsilon_{c_i}}\,dm)\\
&=\lim_{i\to\infty}\frac{1}{c_i}\log(\int_{\sigma^{n_i}(z)=x}e^{c_iR_-^{n_i}(z)+\delta_{c_i}(z)-\delta_{c_i}(x)-n_i\varepsilon_{c_i}}\,dm)\\
&=\lim_{i\to\infty}\frac{1}{c_i}\log(\int_{\sigma^{n_i}(z)=x}e^{c_iR_-^{n_i}(z)-n_i\varepsilon_{c_i}}\,dm)\\
&=\lim_{i\to\infty}\left(\frac{1}{c_i}\log(\int_{\sigma^{n_i}(z)=x}e^{c_iR_-^{n_i}(z)}\,dm)-\frac{n_i\varepsilon_{c_i}}{c_i}\right)=a.
\end{aligned}$$

This shows that any accumulation point have to be equal to zero. □

The first inequality of Theorem 5

Proposition 10. *For any closed set $F \subseteq X$*

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \le \sup_{z\in F} R^\infty_-(z) = -\inf_{x\in F} R^\infty_+(x).$$

Proof. For a fixed k, we have that

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L^{n+k}_{g_c}\chi_F)(x)) = \limsup_{c,n\to\infty} \frac{1}{c}\log(\frac{(L^{n+k}_{g_c}\chi_F)(x)}{(L^{n+k}_{g_c})(x)})$$
$$= \limsup_{c,n\to\infty} \frac{1}{c}\log(\frac{\int_{\sigma^{n+k}(z)=x} e^{cR^{n+k}_-(z)}\chi_F(z)\,dm}{\int_{\sigma^{n+k}(z)=x} e^{cR^{n+k}_-(z)}\,dm})$$
$$= \limsup_{c,n\to\infty} \frac{1}{c}\log(\frac{\int_{\sigma^{n+k}(z)=x} e^{cR^{n+k}_-(z)}\chi_F(z)\,dm}{\int_{\sigma^{n}(y)=x} e^{cR^{n}_-(y)}\,dm})$$
$$= \limsup_{c,n\to\infty} \frac{1}{c}\log(\frac{\int_{\sigma^{n}(y)=x} (e^{cR^{n}_-(y)}\int_{\sigma^{k}(z)=y} e^{cR^{k}_-(z)}\chi_F(z)\,dm)\,dm}{\int_{\sigma^{n}(y)=x} e^{cR^{n}_-(y)}\,dm}).$$

Note, however that

$$\int_{\sigma^k(z)=y} e^{cR^k_-(z)}\chi_F(z)\,dm \le e^{c\sup_{z\in F} R^k_-(z)},$$

then,

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \le \limsup_{c,n\to\infty}\left(\sup_{z\in F} R^k_-(z)\right) = \sup_{z\in F} R^k_-(z).$$

For each k fixed, we have that R^k_- is a continuous function, and $F \subset X$ is a compact set, then, there exist $y_k \in F$, such that, $\sup_{z\in F} R^k_-(z) = R^k_-(y_k)$. Define

$$Y_k := \{y \in F : \limsup_{c,n\to\infty} \frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \le R^k_-(y)\}.$$

Then, Y_k is closed (because R^k_- is a continuous function) and not empty (because $y_k \in Y_k$). Using the fact that $R_- \le 0$ we have

$$Y_1 \supseteq Y_2 \supseteq ...$$

These sets are closed and not empty, then there exist some $x_0 \in \bigcap_{k\ge1} Y_k$. So, for each k:

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L^n_{g_c}\chi_F)(x)) \le R^k_-(x_0).$$

Using the fact that $R_-^k(x_0) \to R_-^\infty(x_0)$, we conclude that

$$\limsup_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^n \chi_F)(x)) \le R_-^\infty(x_0) \le \sup_{z\in F} R_-^\infty(z).$$

□

The second inequality of Theorem 5

Suppose that A is an open set. So, there exists n_0, such that, for $n \ge n_0$, and, $x \in X$, there exists $y \in A$, such that, $\sigma^n(y) = x$. More precisely, given $y = y_1 y_2 ...$ in A, let $\epsilon > 0$, such that, $B(y, \epsilon) \subset A$. Now let n_0 such that $\frac{1}{2^{n_0}} < \epsilon$. If $z \in X$ coincide with $y_1 ... y_{n_0}$ in its first symbols, then $d(z, y) \le \frac{1}{2^{n_0+1}} + \frac{1}{2^{n_0+2}} + ... = \frac{1}{2^{n_0}} < \epsilon$, and, so $z \in A$. We conclude that given $x \in X$, we can get an $y_1 ... y_{n_0} x \in A$.

Lemma 11. *There exist $y_0 \in X$ such that*

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^n \chi_A)(x)) \ge \limsup_{k\to\infty}\left(\sup_{z\in A,\, \sigma^k(z)=y_0} R_-^k(z)\right).$$

Proof. For a fixed $k \ge n_0$, we have

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^{n+k} \chi_A)(x)) = \liminf_{c,n\to\infty} \frac{1}{c}\log(\frac{\int_{\sigma^n(y)=x} e^{cR_-^n(y)}(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\, dm)\, dm}{\int_{\sigma^n(y)=x} e^{cR_-^n(y)}\, dm})$$

$$\ge \liminf_{c,n\to\infty} \frac{1}{c}\log(\inf_{y\in X}\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\, dm)$$

$$= \liminf_{c\to\infty} \inf_{y\in X} \frac{1}{c}\log(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\, dm).$$

Then, we get

$$\liminf_{c,n\to\infty} \frac{1}{c}\log((L_{g_c}^n \chi_A)(x)) \ge \limsup_{k\to\infty}\liminf_{c\to\infty} \inf_{y\in X} \frac{1}{c}\log(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\, dm).$$

Let $y_{c,k}$ be such that

$$\inf_{y\in X} \frac{1}{c}\log(\int_{\sigma^k(z)=y} e^{cR_-^k(z)}\chi_A(z)\, dm) > \frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR_-^k(z)}\chi_A(z)\, dm) - \frac{1}{k}.$$

Then, we have:

$$\liminf_{c,n\to\infty}\frac{1}{c}\log((L^n_{g_c}\chi_A)(x)) \geq \limsup_{k\to\infty}\liminf_{c\to\infty}\frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR^k_-(z)}\chi_A(z)\,dm) - \frac{1}{k}$$

$$= \limsup_{k\to\infty}\liminf_{c\to\infty}\frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR^k_-(z)}\chi_A(z)\,dm). \quad (2)$$

As X is a compact set, let y_0 be an accumulation point of $y_{c,k}$, when $c,k\to\infty$. For k sufficiently large, let $z_k\in A$, such that: $\sigma^k(z_k)=y_0$, and

$$R^k_-(z_k) > \sup_{z\in A,\,\sigma^k(z)=y_0} R^k_-(z) - \frac{1}{2k}.$$

We denote $z_k := x_k...x_1y_0$, $x_i\in M$. We can take $\epsilon>0$ sufficiently small such that the ball $A_{k,\epsilon} := \{a_k...a_1\in M^k : |(a_k...a_1)-(x_k...x_1)|\leq\epsilon\}$ satisfies:
1. $a_k...a_1y_0\in A$,
2. $a_k...a_1y_{c,k}\in A$, for $k,c >> 0$,
3. $R^k_-(a_k...a_1y_0) > \sup_{z\in A,\,\sigma^k(z)=y_0} R^k_-(z) - \frac{1}{k}$.

Then, we get

$$\limsup_{k\to\infty}\liminf_{c\to\infty}\frac{1}{c}\log(\int_{A_{k,\epsilon}} e^{cR^k_-(a_k...a_1y_0)}\,dm)$$

$$\geq \limsup_{k\to\infty}\liminf_{c\to\infty}\left(\sup_{z\in A,\,\sigma^k(z)=y_0} R^k_-(z) - \frac{1}{k} + \frac{1}{c}\log(m(A_{k,\epsilon}))\right)$$

$$= \limsup_{k\to\infty}(\sup_{z\in A,\,\sigma^k(z)=y_0} R^k_-(z)). \quad (3)$$

On the other hand, on $A_{k,\epsilon}$ we have:

$$R^k_-(a_k...a_1y_{c,k}) \geq R^k_-(a_k...a_1y_0) - |R_-|_\theta(\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^k})d(y_{c,k},y_0)$$

$$\geq R^k_-(a_k...a_1y_0) - |R_-|_\theta d(y_{c,k},y_0).$$

Then, we get

$$\limsup_{k\to\infty}\liminf_{c\to\infty}\frac{1}{c}\log(\int_{\sigma^k(z)=y_{c,k}} e^{cR^k_-(z)}\chi_A(z)\,dm)$$

$$\geq \limsup_{k\to\infty}\liminf_{c\to\infty}\frac{1}{c}\log(\int_{A_{k,\epsilon}} e^{cR^k_-(a_k...a_1y_{c,k})}\,dm)$$

$$\geq \limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c} \log\Big(\int_{A_{k,\epsilon}} e^{cR_-^k(a_k\ldots a_1 y_0) - c|R_-|_\theta d(y_{c,k},y_0)}\, dm\Big)$$

$$= \limsup_{k\to\infty} \liminf_{c\to\infty} \frac{1}{c} \log\Big(\int_{A_{k,\epsilon}} e^{cR_-^k(a_k\ldots a_1 y_0)}\, dm\Big). \tag{4}$$

Using (2), (4) and (3), we finish the proof. □

Now we fix the point y_0 given above. The next result is basically contained in the proof of proposition 5 in [6].

Lemma 12. *Let p be a point on the support of μ_∞. Let y_n a sequence satisfying $\sigma(y_n) = y_{n-1}, n = 1, 2, 3, \ldots,$ and, $0 = R_-(y_1) = R_-(y_2) = \ldots$ (it follows from the property of the calibrated subaction). Then, p is a accumulation point of $\{y_n\}$.*

Proof. Let B be the set of accumulation points of $\{y_n\}$. B is closed and $\sigma(B) = B$. Then there exists a invariant probability ν with support on B. The inclusion $B \subseteq X$ implies the existence of an extension of ν to X by the rule: $\nu(\phi) := \nu(\phi.\chi_B)$. Using the fact that R_- is a continuous function, and, that $R_-(y_n) = 0, n = 1, 2, \ldots,$ we conclude that $\chi_B.R_- = 0$. Therefore, $\nu(R_-) = 0$, and then, $\nu = \mu_\infty$. From this we get that the support of μ_∞ is contained on B. □

The next lemma follows the same reasoning of Lemma 18 in [30]:

Lemma 13. *If $R_-^\infty(z) > -\infty$, then the family of probabilities ν_n (also called empirical measures), given by $\phi \to \frac{1}{n}\sum_{j=0}^{n-1} \phi(\sigma^j(z))$, converges to μ_∞ weakly*, when $n \to \infty$.*

Proof. Note that any measure which in accumulation point of ν_n is an invariant probability. We are going to show that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\sigma^j(z)) \geq \mu_\infty(f).$$

Let $M = R_-^\infty(z)$. Then, for each n we have $R_-^n(z) \geq M$, so:

$$V(z) - V(\sigma^n(z)) - n\mu_\infty(f) + \sum_{j=0}^{n-1} f(\sigma^j(z)) = \sum_{j=0}^{n-1} R_-(\sigma^j(z)) = R_-^n(z) \geq M.$$

Then, we get

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\sigma^j(z)) \geq \frac{M}{n} - \frac{2|V|_\infty}{n} + \mu_\infty(f).$$

Finally, taking $\liminf_{n\to\infty}$ in the above expression we get our claim. □

Corollary 14. *If $R_-^\infty(z) > -\infty$, and, $p \in supp(\mu_\infty)$, then p is an accumulation point of $\sigma^n(z)$.*

Proof. Let $p \in supp(\mu_\infty)$, and $\varepsilon > 0$. Consider the ball $B(p,\varepsilon) := \{x \in X : d(x,p) < \varepsilon\}$. Using the fact that $p \in supp(\mu_\infty)$, we have that $\mu_\infty(B(p,\varepsilon)) > 0$. Therefore, by the above lemma, we have that $\{\sigma^n(z)\}$ is in this ball for infinite values of n. □

Lemma 15.

$$\sup_{z\in A} R_-^\infty(z) \le \limsup_{k\to\infty}(\sup_{z\in A,\,\sigma^k(z)=y_0} R_-^k(z)).$$

Proof. We fix a point $p \in supp(\mu_\infty)$, and, we denote $p = p_1p_2....$ For $n \ge n_0$, there exists $y \in A$, such that $\sigma^n(y) = p$. Note that $R_-^\infty(y) = R_-^n(y) > -\infty$. Therefore, $\sup_{z\in A} R_-^\infty(z) > -\infty$. Let $z_0 \in A$ be such that, $R_-^\infty(z_0) > -\infty$, and, denote $z_0 = x_1x_2....$ Given $t \in \mathbb{N}$, let $n(t)$ be such that, $d(\sigma^{n(t)}(z_0), p) \le \frac{1}{2^t}$, and moreover, such that, the choice $x_1x_2...x_{n(t)}$, determines that $z_0 \in A$ (open).

From Lemma 12 there exists a pre-image of y_0 (we assume it is a point $y_{l(t)}$ of the form $a_{l(t)}...a_1(y_0)$), such that, $R_-^{l(t)}(y_{l(t)}) = 0$, and, $d(y_{l(t)}, p) < \frac{1}{2^t}$. Define $z(t)$ by

$$z(t) := x_1...x_{n(t)}a_{l(t)}...a_1(y_0).$$

Then,

$$\begin{aligned}\sup_{z\in A:\sigma^{l(t)+n(t)}(z)=y_0} R_-^{l(t)+n(t)}(z) &\ge R_-^{l(t)+n(t)}(z(t)) \\ &= R_-^{n(t)}(z(t)) + R_-^{l(t)}(y_{l(t)}) = R_-^{n(t)}(z(t)) \\ &\ge R_-^{n(t)}(z_0) - |R_-|_\theta 2(\frac{1}{2^t} + \frac{1}{2^{t+1}} + ... + \frac{1}{2^{t+n(t)}}) \\ &\ge R_-^{n(t)}(z_0) - \frac{|R_-|_\theta}{2^t} \ge R_-^\infty(z_0) - \frac{|R_-|_\theta}{2^t}.\end{aligned}$$

When, $t \to \infty$, we get

$$\limsup_{k\to\infty}(\sup_{z\in A,\,\sigma^k(z)=y_0} R_-^k(z)) \ge \limsup_{t\to\infty} \sup_{z\in A:\sigma^{l(t)+n(t)}(z)=y_0} R_-^{l(t)+n(t)}(z)$$

$$\ge R_-^\infty(z_0).$$

Using the fact that z_0 is arbitrary, and satisfies $R_-^\infty(z_0) > -\infty$, we finish the proof. □

From this result and lemma 11 we get:

Proposition 16.

$$\liminf_{c,n\to\infty} \frac{1}{c} \log((L_{g_c}^n \chi_A)(x)) \geq \sup_{z\in A} R_-^\infty(z) = -\inf_{z\in A}(R_+^\infty(z)).$$

This finish the proof of Theorem 5.

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