ADVANCES IN MATHEMATICS **101**, 133–165 (1993)

The Zeta Function, Non-differentiability of Pressure, and the Critical Exponent of Transition

A. O. LOPES*, †

Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

The main purpose of this paper is to analyze the lack of differentiability of the pressure and, from the behaviour of the pressure around the point of non-differentiability, to derive an asymptotic formula for the number of periodic orbits (under certain restrictions related to the norm of the periodic orbit) of a dynamical system. This kind of results is analogous to the well known Theorem of Distribution of Primes of "Introduction to Analytic Number Theory" (T. M. Apostol, 1976, Springer-Verlag, New York/Berlin). This result follows from analysis of the dynamic zeta function and Tauberian theorems. We introduce a functional equation relating the pressure and the Riemann zeta function, and this equation plays an essential role in the proof of our results. We can say, in general terms, that the result presented here extends some well known results obtained for expanding dynamical systems to a certain class of non-expanding dynamical systems. From another point of view, we can say that we are analyzing thermodynamic formalism for non-Hölder functions (or for functions not in the class F_θ considered by D. Ruelle, W. Parry, and M. Pollicot). As an example of the results presented here, we show that for the Manneville–Pomeau map $f\colon [0,1] \to [0,1]$, given by $f(x) = x + x^{1+s} \pmod 1$, where s is a positive real constant, $0.5 < s < 1$, the pressure $p(t) = \sup_{v \in M(f)} \{h(v) - t \int \log |f'(x)|\, dv(x)\}$ is such that

$$p(t) \approx \begin{cases} h(\mu)(1-t) + B(1-t)^{1/s}, & \text{for} \quad t < 1 \\ 0, & \text{for} \quad t \geqslant 1, \end{cases}$$

where B is a constant and $h(\mu)$ is the entropy of the Bowen–Ruelle–Sinai measure. The above result is an example of a first order phase transition. In this case, the pressure is not differentiable at $t = 1$, and s^{-1} will be the critical exponent of transition. Some of the results can also be seen as a result in number theory for partitions with weights. We give a proof of a result of B. Felderhof and M. Fisher (1970, *Ann. Phys. (N.Y.)* **58**, 176–281; 1967, *Physica* **3**, 255–283) concerning the critical exponent of transition in the setting of thermodynamic formalism.

INTRODUCTION

We consider σ the shift in two symbols in the one-dimensional lattice $\mathbb{N}$. The Bernoulli space $\{0, 1\}^{\mathbb{N}}$ is denoted by Σ.

* Partially supported by the AFOSR.

† Permanent address: Instituto de Matematica, URFGS, Porto Alegre, Brasil.

0001-8708/93 $9.00

Our results can be stated in more general terms, but in order to avoid a more complicated notation, we are restricted to consider here such a dynamical system. We also consider later the Manneville–Pomeau map (see Section 3).

Consider g a continuous real valued function on the Bernoulli space Σ. If g is in the class F_θ (see Section 1) or Hölder-continuous, the variational problem

$$\sup_{\tilde{v} \in M(\sigma)} \left\{ h(\tilde{v}) - \int g(\mathbf{x})\, d\tilde{v}(\mathbf{x}) \right\}$$

has a unique solution [31], as is very well known.

Here $M(\sigma)$ is the set of σ-invariant probabilities, and $h(\tilde{v})$ is the entropy of the measure $\tilde{v}$ (see [22] for references).

We denote by $P(g)$ the above supremum value of the variational problem. We call $P(g)$ the pressure associated with the potential g. We also denote by $\tilde{v}_g$ the equilibrium state (the unique solution of the variational problem) associated with g. That is, $P(g) = h(\tilde{v}_g) + \int g(\mathbf{x})\, d\tilde{v}_g(\mathbf{x})\}$.

It is also known that, when g is in the class F_θ, then $P(g)$ is differentiable in g, and the Ruelle zeta function

$$\zeta_g(z) = \exp \sum_{n=0}^{\infty} \frac{z^n}{n} \left(\sum_{\substack{J \text{ periodic} \\ \text{orbit of} \\ \text{period } n}} e^{\sum_{j=0}^{n-1} g(\sigma^j(J))} \right)$$

is meromorphic and has a simple pole at $z = e^{P(g)}$ [23, 26, 27, 31]. A related result is shown in Section 1. In Section 2 we consider a different kind of zeta function introduced by W. Parry [24]. We call this function the dynamic zeta function.

We analyze here a class of potentials g (not in the class F_θ) that were previously considered by F. Hofbauer [13]. For some of these potentials g, equilibrium states are not unique (see [13]).

Now consider g a fixed potential. We introduce an external parameter t (as in [17, 19]) and analyze the one parameter family of the equilibrium state $\tilde{v}_{tg}$ for the potentials of form tg.

In [19] we analyze phase transition properties of the above family of potentials in the setting of the generalized dimension of the maximal measure. An announcement of the results of the present paper appeared in 1990 in the mentioned paper.

In our considerations here, we denote $\mu_t = \tilde{v}_{tg}$. The probabilities μ_t are unique and will change in a continuous fashion, until we reach a transition value of the parameter $t = t_0$, where μ_t is not continuous in t.

For g in the class F_θ, the above property cannot occur. That is, equilibrium states always change in a continuous fashion with t [26, 32].

In other words, phase transitions do not occur for g in the class F_θ [25, 26, 32].

For the class of potentials g that we consider here, at the transition value $t = t_0$, there exist two possibilities depending on g: two equilibrium states or one equilibrium state. Following the usual terminology of statistical mechanics, we say that in the first case we have a first order phase transition and in the second case a second order transition. In the first case there exists lack of differentiability of the function $p(t) = P(tg)$ at $t = t_0$. In the second case $p(t)$ is differentiable in $t = t_0$ (and also for all values of t) but it is not C^∞. The function $p(t)$ is sometimes called free energy associated with g [6, 17, 19]. The important point here is that depending on the class H_γ (see Section 1) to which the potential g belongs, we are able to express the kind of lack of differentiability one has for $p(t)$ at $t = t_0$.

In concrete terms we have the following result: there exists $\Delta > 0$ such that

$$p(t) = P(tg) \approx \begin{cases} A(t_0 - t) + B(t_0 - t)^\Delta, & t < t_0 \\ 0, & t \geqslant t_0, \end{cases}$$

where A and B are constants (see Theorem 1).

The value Δ depends on the function g (see the definition of H_γ, Section 1) and is called the critical exponent of transition.

We mentioned, some paragraphs above, the existence of two possibilities: two equilibrium states or one equilibrium state. In the first case, the constant A above is different from zero; in the second case it is zero.

We would like to point out the use of the above notation: we say that $a_n \sim b_n$, $n \in \mathbb{N}$, when $\exists C_1, C_2 > 0$ such that $C_1 < a_n/b_n < C_2$, when n goes to ∞. We also use the notation $P(t) \approx A(t_0 - t) + B(t_0 - t)^\Delta$ (or sometimes $a_n \approx b_n$), which means that the quotient

$$\lim_{t \to \infty} \frac{P(t)}{A(t_0 - t) + B(t_0 - t)} = 1 \qquad \left(\text{when } \lim_{n \to \infty} \frac{a_n}{b_n} = 1\right).$$

The symbols $\sim$ and $\approx$ used in this paper should be consider as a device for simplification of the notation, and not as a lack of mathematical precision. We also use the capital letters A, B, C, D and C_1, C_2 for general constants that can change from one place to another.

Now that we have explained our notation, let us continue with the explanation of the results that we show here.

From the above analytical type of lack of differentiability, we will be able to derive a result about the asymptotic growth number of periodic orbits (see Theorem 2). We also analyze in Section 3 a related problem. For a given fixed value of $s>0$, consider the map from the interval $[0, 1]$ in itself given by

$$f(x)=x+x^{1+s} \qquad (\text{mod } 1).$$

This map is sometimes called the Manneville–Pomeau map [4, 37]. This map is not expanding. The fixed indifferent point 0 has derivative 1 and the analytical expression of $f'(x)$ for x around zero is $f'(x)=1+(s+1)x^s$. We refer the reader to [4] for an interesting explanation of how this map was obtained from Poincaré sections related to the Lorenz attractor.

In Section 3 we consider the variational problem

$$P(t)=\sup_{\tilde{v}\in M(f)}\left\{h(\tilde{v})-t\int \log|f'(x)|\,d\tilde{v}(x)\right\},$$

where $M(f)$ is the set of invariant measures of f.

Using the preceding results obtained in Sections 1 and 2, we show (depending on s) the type of lack of analyticity of the function $P(t)$ around $t=1$. (See Theorem 3 in Section 3.)

In order to have a correct perspective of the results presented here in Sections 1, 2, and 3, we believe it is now worthwhile to give a brief historical description of the evolution of the analysis of the physical problems related to the lack of differentiability of the free energy $p(t)=P(tg)$, also known as phase transition problems.

An example of this kind of problems is the sudden magnetization that occurs at low temperature for some ferromagnetic materials.

First we point out that we are analyzing equilibrium states (Gibbs states) in the setting of entropy and pressure in the sense of thermodynamic formalism [31]. This theory was devised by R. Bowen, Ya. Sinai, and D. Ruelle and gives a rigorous mathematical formulation to several problems (no phase transition) in statistical mechanics in one-dimensional lattices [26, 27, 31, 36]. The classical references of thermodynamic formalism do not consider problems of phase transition in general. The procedure most physicists use to analyze statistical mechanics in one-dimensional lattices is different from that mentioned previously (see [19] for a brief explanation of the differences). In fact, the mathematical theory proposed by Bowen, Sinai, and Ruelle is much more recent than the theory used by most physicists [31].

An important result of D. Ruelle shows that by choosing the correct potential g in the lattice $\mathbb{Z}$, one can recover the exact Gibbs state that physicists have long known (see [31] for references).

The pressure of thermodynamic formalism is minus the pressure of classical statistical mechanics used by most physicists.

Recall the critical exponent of transition Δ in the local expression of $P(t)$ around $t=t_0$ mentioned above. This Δ appears in the thermodynamic formalism setting, and there is also an analogous Δ for the free energy as considered by physicists. In fact, the Δ of the physicists came first, and we are just formalizing in the context of thermodynamic formalism the Δ that they have measured experimentally.

It was a long time before an explanation in physical terms for the value Δ measured by physicists in phase transition problems was given. B. Felderhof and M. Fisher in [9, 11], 1967, 1970, made a major contribution in statistical physics by presenting an explanation for the phenomena (in the setting used by physicists). We are not able to understand the nature of this explanation.

We also point out the very important contribution in the physics literature of the papers of Wang [37, 38] (see also [4, 5, 7, 8, 12, 33]). These paper analyze the free energy $P(t)$ in the setting of thermodynamic formalism for a kind of piecewise linear version of the Manneville–Pomeau map. Such maps are supposed to model and explain transitions from laminar to intermittent behaviour (see [37, 4] for explanations of a physical nature related to renormlization group, noise, universality, etc.).

The expression of $P(t)$ for t around t_0 (the transition value) that we rigorously prove here for the Manneville–Pomeau map and for the shift was predicted by Wang [37, 38] for the piecewise linear approximation of the Manneville–Pomeau map.

The essential tools with which to formalize the results stated in [4, 37, 38] are the Ruelle–Perron–Frobenius operator techniques. We use some nice theorems obtained for such operators by F. Hofbauer [13], in order to obtain rigorous mathematical proofs of the above-mentioned results stated in [37, 38]. We do not assume a Markow property as in [37, 38].

Our proof in Sections 1, 2, and 3 has no intersection with the ideas of Wang [37, 38]. The proof in Section 1 is more analytic, and the dynamic part is contained in the use of F. Hofbauer's results.

In Section 4 we give another demonstration (again using the Ruelle–Perron–Frobenius operator techniques of F. Hofbauer) of the same Theorem 1 obtained in Section 1, but this time we give rigorous mathematical proofs of the beautiful ideas of Wang. This proof is purely dynamic and follows from considerations of random walks in the lattice $\mathbb{N}$ and recurrent events.

The results obtained here, together with considerations of Wang [38], formalize in a rigorous way interesting considerations of $1/f$ noise in [39] (see the end of Section 4).

We also mention that in [17, 20] the phenomenon of phase transition is related to the fact that sometimes the critical points are eventually periodic. In another way, the phase transition presented in [19] and also in this paper is related to indifferent (or neutral) periodic orbits.

As is well known, the two main obstructions of the expansion of one-dimensional maps are the possibilities of the existence of critical points and the existence of indifferent periodic points in the non-wandering set.

It follows from the above considerations that either of the two cases can produce lack of differentiability for the pressure. The lack of differentiability in any case is not very serious: we have limits for the derivative at the left and right sides of the transition value. In this situation, large deviation results also apply [6, Chap. 6] and the theorems obtained in [16, 17] (with obvious modifications) can also be obtained for the Manneville–Pomeau map.

In [19], a detailed expository explanation of "generalized dimension transitions for the maximal entropy measure" is presented, and the relation of this mathematical setting to problems in physics is explained to a more general audience. For example, a Dirac delta in a fixed point should be seen as a magnetization (see [19]). Antiferromagnetic arrangements also occur in the transition value for some models given by polynomial maps (see [20]).

Some of the statements in [19] are referred to the mathematical proofs given here.

In general terms, the two settings "pressure" and "generalized dimension spectrum of the maximal entropy measure" are equivalent points of view of the same problem [16–19]. They are related by Legendre transforms in the same way as Lagrangian and Hamiltonian mechanics [19].

In Appendix B in [29], it is mentioned that the Blaschke product $f(z) = ((3z+1)/(3+z))^2$ restricted to the unit circle has no f-invariant probability equivalent to the arc length. The point $p = 1$ is fixed and indifferent (that is, $|f'(1)| = 1$). The Ruelle–Perron–Frobenius operators do not have all the good properties of the expanding case, as is mentioned in [29]; however, the considerations of Section 3 here will apply to this situation. We will not have exponential decay of correlation, as we see at the end of Section 4. See also [30, 41] for results related to the above considerations.

In [14], some of the ideas presented here are applied to geodesics in a manifold of constant negative curvature.

1. The Functional Equation and the Critical Exponent of Transition

We consider a slight modification of the examples mentioned in [13, 19].

Denote $M_0 = \{(0, x_1, x_2, ...) \mid x_i \in \{1, 0\},\ i \geqslant 1\}$ and $M_k = \{\mathbf{x} \in \Sigma \mid x_i = 1$ for $0 \leqslant i \leqslant k-1,\ x_k = 0\}$ for $k = 1, 2, ...$ We denote, as usual, ζ the Riemann zeta function [3].

For each value of $\gamma \in (1, \infty)$ consider the scalar-valued function

$$g(\mathbf{x}) = a_k = -\gamma \log\left(\frac{k+1}{k}\right), \qquad \mathbf{x} \in M_k, k \neq 0,$$

$$g(\mathbf{x}) = a_0 = -\log(\zeta(\gamma)), \qquad x \in M_0$$

and

$$g(111\cdots) = 0.$$

Note that $\lim_{k\to\infty} a_k = 0$ (see [13, (a), p. 238]). In fact, the results presented in Sections 1 and 2 apply for a more general class of potentials g.

We say that g belong to the class H_γ if $\exists C_1, C_2 > 0$ such that $C_1 a_k \leqslant g(\mathbf{x}) \leqslant C_2 a_k$ for all $k \in \mathbb{N}$, $\mathbf{x} \in M_k$, and γ as above. The results presented here can be extended to potentials g in H_γ.

It is worthwhile to have a visualization of the action of σ in the partition M_k, $k \in \mathbb{N}$, in an equivalent setting. The map l from $[0, 1]$ to itself, such that $l(x) = 2x$ for $x \in [0, \frac{1}{2})$ and $l(x) = 2(x - \frac{1}{2})$ for $x \in [\frac{1}{2}, 1]$, is conjugated with σ. In Fig. 1 we show the graph of l and the intervals M_k. Note that $l(M_k) = M_{k-1}$ and l^k is a diffeomorphism from M_k to M_0, for all $k \in \mathbb{N}$.

Consider $s_k = a_0 + a_1 + \cdots + a_k$, $k \in \mathbb{N}$, as in [13].

The sum

$$\sum_{k=0}^{\infty} e^{s_k} = e^{a_0}\left(1 + \sum_{k=1}^{\infty} a_1 + \cdots + a_k\right)$$

$$= \zeta(\gamma)^{-1}\left(1 + \sum_{k=1}^{\infty} (k+1)^{-\gamma}\right)$$

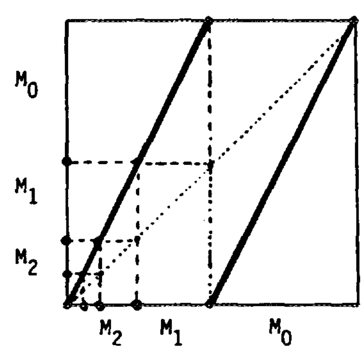

FIGURE 1

is therefore equal to 1 (see [13, (b), p. 238]). Note that

$$\sum_{k=0}^{\infty} (k+1)\, e^{s_k} = e^{a_0} \sum_{k=1}^{\infty} (k+1)^{1-\gamma}$$

is finite for $\gamma \in (2, 3)$ and infinite for $\gamma \in (1, 2]$ (see [13, condition (c), p. 238]).

From now on we consider a fixed value of γ. When differences between the value of γ ($\gamma \in (1, 2]$ or $\gamma \in (2, 3)$) imply different results, we point out the separate result in each situation.

Note that g depend on γ and we write g_γ when we want to emphasize this fact.

For each fixed value of γ consider $g(\mathbf{x})$ as above and also for $t \in \mathbb{R}$ the function $p(t) = P(tg)$, where the last expression is the pressure of the scalar function $tg(\mathbf{x})$ in the variable $\mathbf{x}$.

For a continuous function $f\colon \Sigma \to \mathbb{R}$ and $n \geqslant 0$ we define

$$\operatorname{var}_n f = \sup\{ |f(\mathbf{x}) - f(\mathbf{y})| \mid x_i = y_i,\ i \leqslant n \}.$$

If one considers the space $F_\theta = \{ f \mid f$ continuous, $\operatorname{var}_n f \leqslant C\theta^n$; $n \in \mathbb{N}$ for some $C > 0\}$, where θ is fixed, then for $f \in F_\theta$, the real function of $t \in \mathbb{R}$, $p(t) = P(tf)$, is infinitely differentiable, and for each $t \in \mathbb{R}$ the equilibrium measure for the variational problem

$$p(t) = P(tf) = \sup_{v \in M(\sigma)} \left\{ h(v) + t \int f(\mathbf{x})\, dv(\mathbf{x}) \right\}$$

is unique [27]. Therefore, phase transitions do not occur [16, 25, 32].

The functions g that we consider here are not in F_θ for $\theta \in \mathbb{R}$.

Following F. Hofbauer, we consider $C(\Sigma)$ the Banach space of real valued continuous functions with supremum norm $\|\cdot\|$. We define, for each $\varphi \in C(\Sigma)$, the operator $\mathscr{L}_\varphi$ on $C(\Sigma)$ by

$$\mathscr{L} f(\mathbf{x}) = \sum_{\mathbf{y} \in \sigma^{-1}(\mathbf{x})} e^{\varphi(\mathbf{y})} f(\mathbf{y}).$$

Then

$$\mathscr{L}_\varphi^m f(\mathbf{x}) = \sum_{\mathbf{y} \in \sigma^{-m}(\mathbf{x})} \exp\left(\sum_{i=0}^{m-1} \varphi(\sigma^i(\mathbf{y})) \right) f(\mathbf{y}).$$

Denote by $M(\Sigma)$ the space of probabilities on Σ. We say that φ satisfies the RPF condition (Ruelle–Perron–Frobenius condition) if there are $\lambda > 0$, $h \in C(\Sigma)$ with $h > 0$, and $v \in M(\Sigma)$ for which

$$\mathscr{L}_\varphi h = \lambda h, \qquad \mathscr{L}_\varphi^* v = \lambda v, \qquad \int h(\mathbf{x})\, dv(\mathbf{x}) = v(h) = 1$$

and

$$\lim_{m \to \infty} \|\lambda^{-m} \mathscr{L}_\varphi^m f - v(f) h\| = 0 \qquad \text{for all} \quad f \in C(\Sigma).$$

The measure μ defined by $\mu(f) = v(hf)$ is called the RPF measure, and it will belong to $M(\sigma)$.

When φ satisfies the RPF condition, then $\mu = vh$ is the unique equilibrium state for φ (see Theorem 1.1 of [13]); that is, μ is the only solution to the variational problem

$$P(\varphi) = \sup_{v \in M(\sigma)} \left\{ h(v) + \int \varphi(\mathbf{x})\, dv(\mathbf{x}) \right\}.$$

Note that μ is an invariant probability, but v not necessarily. This point will be important later. Note also that from the theorem on page 226 in [13], it follows that, for each $t < 1$, $\sum_{k=0}^{\infty} e^{ts_k} > 1$, and therefore tg satisfies the RPF condition.

For a fixed γ we denote μ_t, $t < 1$, the unique equilibrium state for tg_γ. We denote by v_t the associated probability in $M(\Sigma)$ (for $t < 1$), as above.

The problem is, of course, with the value $t = 1$, because for $t > 1$, it follows from [13, Lemma 4.6, p. 235] that the Dirac $\delta_{111\ldots}$ is equilibrium state for tg.

Now we must make a distinction for different values of γ. Suppose $\gamma \in (1, 2]$; then $\sum_{k=1}^{\infty} (k+1) e^{s_k} = \infty$, and for $t = 1, g$ has a unique equilibrium state (see [13, table on p. 239]), namely $\delta_{111\ldots}$.

If $\gamma \in (2, 3)$, then $\sum_{k=1}^{\infty} (k+1) e^{s_k} < \infty$ and there exist two equilibrium states. One equilibrium state in $\delta_{111\ldots}$ and the other is denoted by μ.

In any case $p(t) = 0$ for $t \geq 1$.

Remark 1. Note the important point that all results presented here depend on $e^{s_k} \sim k^{-\gamma}$ and not on the exact value of a_k, $k \in \mathbb{N}$.

We want to show that for t close to the left of the value 1 we have the following behaviour of $p(t)$:

THEOREM 1. *Under the above definitions we have two possibilities*:

(a) $1 < \gamma < 2$; *then for* $t \leq 1$, $t \to 1$,

$$p(t) = \left(\frac{\zeta(\gamma) \log \zeta(\gamma) - \gamma \zeta'(\gamma)}{-\Gamma(1-\gamma)} \right)^{1/(\gamma-1)} (1-t)^{1/(\gamma-1)} + \textit{high order terms},$$

or

(b) $2<\gamma<3$; *then for* $t \leqslant 1$, $t \to 1$

$$p(t)=\frac{\zeta(\gamma)\log\zeta(\gamma)-\gamma\zeta'(\gamma)}{\gamma\zeta'(\gamma-1)}(1-t)+A(1-t)^{\gamma-1}(1+o(1)).$$

In the last case, it follows that the entropy of μ (equilibrium state for g) is

$$\frac{\zeta(\gamma)\log\zeta(\gamma)-\gamma\zeta'(\gamma)}{\gamma\zeta'(\gamma-1)}.$$

The case $\gamma>3$ is also analyzed at the end of our proof of Theorem 1. The formulas are more complex in this case.

Proof. The above result follows from the following functional equation satisfied by $p(t)$: for $t<1$,

$$\zeta(\gamma)^t=\sum_{n=1}^{\infty}\frac{e^{-np(t)}}{n^{\gamma t}}.$$

From this equation it also follows that $p(t)$ is real analytic for $t<1$.

Remark 2. The value of the entropy of μ above can be directly obtained by taking the derivative of the functional equation and evaluating at 1.

Let us first show the functional equation. Recall from page 226 of [13] that $\nu_{k,t}=\lambda_t^{-(k+1)}e^{ts_k}$, $k\neq 0$, and $\nu_{0,t}=e^{ta_0}\cdot\lambda_t^{-1}$, where $\log\lambda_t=p(t)$. We also have $\sum_{k=0}^{\infty}\nu_{k,t}=1$, and therefore,

$$\begin{aligned}1&=e^{ta_0}e^{-p(t)}+e^{ta_0}\sum_{k=1}^{\infty}e^{-(k+1)p(t)}(k+1)^{-t\gamma}\\&=e^{ta_0}\left(\sum_{n=0}^{\infty}e^{-(k+1)p(t)}(k+1)^{-t\gamma}\right).\end{aligned}$$

As $e^{a_0}=\zeta(\gamma)^{-1}$, the functional equation is shown to be true.

Now we show property (a). First we need a lemma.

LEMMA 1. *If* $r>-1$, *then*

$$\sum_{n=1}^{\infty}n^r e^{-n\mu}\sim\frac{\Gamma(1+r)}{\mu^{1+r}}\qquad as\quad \mu\searrow 0.$$

The proof of the lemma follows easily by approximating the sum by an integral.

Now we proceed with the proof of (a).

Set $t=1-\xi$ and $p(t)=\delta$ (δ will be small because $p(1)=0$). First consider the Taylor expansion

$$\zeta(\gamma)^{1-\xi}=\zeta(\gamma)-\zeta(\gamma)\log\zeta(\gamma)\xi+O(\xi^2).$$

Consider now

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{\gamma t}}e^{-n\delta}&=\zeta(\gamma t)-\sum_{n=1}^{\infty}n^{-\gamma t}(1-e^{-n\delta})\\&=\zeta(\gamma t)-\sum_{n=1}^{\infty}n^{1-\gamma(1-\xi)}\left(-\frac{1}{n}\right)(e^{-n\delta}-1)\\&=\zeta(\gamma t)-\sum_{n=1}^{\infty}n^{1-\gamma+\gamma\xi}\int_0^{\delta}e^{-n\mu}\,d\mu\\&=\zeta(\gamma t)-\int_0^{\delta}\left(\sum_{n=1}^{\infty}n^{1-\gamma+\gamma\xi}e^{-n\mu}\right)d\mu.\end{aligned}$$

The last equality follows from the positivity of the summands.

Now using the last lemma above, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{e^{-n\delta}}{n^{\gamma t}}&=\zeta(\gamma t)-\int_0^{\delta}\frac{\Gamma(2-\gamma+\gamma\xi)}{\mu^{2-\gamma+\gamma\xi}}(1+o(1))\,d\mu\\&=\zeta(\gamma(1-\xi))-\frac{\Gamma(2-\gamma+\gamma\xi)(1+o(1))}{\gamma-1-\gamma\xi}\delta^{\gamma-\gamma\xi-1}\\&=\zeta(\gamma)-\gamma\xi\zeta'(\gamma)+O(\xi^2)+\Gamma(1-\gamma+\gamma\xi)(1+o(1))\,\delta^{\gamma-\gamma\xi-1},\end{aligned}$$

since $\Gamma(2-\gamma+\gamma\xi)=(1-\gamma+\gamma\xi)\,\Gamma(1-\gamma+\gamma\xi)$. Hence

$$\sum_{n=1}^{\infty}\frac{e^{-n\delta}}{n^{\gamma t}}=\zeta(\gamma)-\gamma\zeta'(\gamma)\xi+O(\xi^2)+\Gamma(1-\gamma)(1+o(1))\,\delta^{\gamma-\gamma\xi-1}.$$

Now from the functional equation

$$\begin{aligned}&\zeta(\gamma)-\zeta(\gamma)\log\zeta(\gamma)\xi+O(\xi^2)\\&\quad=\zeta(\gamma)^{1-\xi}=\zeta(\gamma)^t=\sum_{n=1}^{\infty}\frac{1}{n^{\gamma t}}e^{-np(t)}=\sum_{n=1}^{\infty}\frac{1}{n^{\gamma t}}e^{-n\delta}\\&\quad=\zeta(\gamma)-\zeta'(\gamma)\,\gamma\xi-\Gamma(1-\gamma)(1+o(1))\,\delta^{\gamma-\gamma\xi-1}+o(\xi^2),\end{aligned}$$

we conclude that

$$\delta=\left[\xi\,\frac{(\zeta(\gamma)\log\zeta(\gamma)-\zeta'(\gamma)\gamma)}{\Gamma(1-\gamma)}\,(1+o(1))\right]^{1/(\gamma-1-\gamma\xi)}.$$

This is the end of the proof of (a).

Now let us show (b). Suppose $2<\gamma<3$. Again consider $t=1-\xi$ and $p(t)=\delta$.

From the identity

$$e^{-n\delta}=1-n\delta+n^2\int_0^\delta(\delta-\mu)\,e^{-n\mu}\,d\mu$$

we deduce

$$\sum_{n=1}^{\infty}n^{-\gamma t}\,e^{-n\delta}=\zeta(\gamma t)-\delta\zeta(\gamma t-1)+\int_0^\delta(\delta-\mu)\left(\sum_{n=1}^{\infty}n^{2-\gamma t}\,e^{-n\mu}\right)d\mu.$$

From the lemma (note that $2-\gamma t>-1$ for t close enough to 1), we have

$$\begin{aligned}\int_0^\delta(\delta-\mu)&\left(\sum_{n=1}^{\infty}n^{2-\gamma t}\,e^{-n\mu}\right)d\mu\\&\sim\Gamma(3-\gamma t)\int_0^\delta\frac{\delta-\mu}{\mu^{3-\gamma t}}\,d\mu\\&=\Gamma(3-\gamma t)\left(\frac{\delta^{\gamma t-1}}{\gamma t-2}-\frac{\delta^{\gamma t-1}}{\gamma t-1}\right)=\Gamma(1-\gamma t)\,\delta^{\gamma t-1}\end{aligned}$$

as $\delta\to 0$. Hence for $t=1-\varepsilon$ with ε and δ small we have

$$\begin{aligned}\sum_{n=1}^{\infty}n^{-\gamma t}\,e^{-n\delta}&=\zeta(\gamma)-\gamma\zeta'(\gamma)\varepsilon+O(\varepsilon^2)-\delta\zeta(\gamma-1)\\&\quad+O(\delta\varepsilon)+(\Gamma(1-\gamma)+o(1))\,\delta^{\gamma t-1}.\end{aligned}$$

On the other hand,

$$\zeta(\gamma)^t=\zeta(\gamma)\,e^{-\varepsilon\log\zeta(\gamma)}=\zeta(\gamma)-\varepsilon\zeta(\gamma)\log\zeta(\gamma)+O(\varepsilon^2).$$

Equating these two expressions gives $\delta=A_1\varepsilon+(C+o(1))\,\varepsilon^{\gamma-1}$ with

$$A_1=\frac{\zeta(\gamma)\log\zeta(\gamma)-\gamma\zeta'(\gamma)}{\zeta(\gamma-1)},\qquad C=\frac{\Gamma(1-\gamma)}{\zeta(\gamma-1)}\,A_1^{\gamma},$$

as claimed.

Remark 3. In the same way, using the identity

$$e^{-n\delta}=\sum_{j=0}^{m-1}\frac{(-1)^j}{j!}n^j\delta^j+\frac{(-1)^m}{(m-1)!}n^m\int_0^\delta(\delta-\mu)^{m-1}e^{-n\mu}\,d\mu,$$

we find for $m<\gamma t<m+1$ $(m\geqslant 2)$ the expansion

$$\begin{aligned}\sum_{n=1}^{\infty}n^{-\gamma t}e^{-n\delta}=\sum_{j=0}^{m-1}&\frac{(-1)^j}{j!}\zeta(\gamma t-j)\,\delta^j\\&+(\zeta(1-\gamma t)+o(1))\,\delta^{\gamma t-1}\qquad(\delta\to 0)\end{aligned}$$

and hence for $t\to 1$ the expansion

$$\begin{aligned}p(t)=A_1(1-t)&+A_2(1-t)^2+\cdots+A_{m-1}(1-t)^{m-1}\\&+(C+o(1))(1-t)^{\gamma-1}\qquad(m<\gamma<m+1,\ t\to 1)\end{aligned}$$

with certain constants $A_1, A_2, ..., A_{m-1}$ and C, the formulas for A_1 and C being the same as those given above. In particular, $C\neq 0$ and $C\to\infty$ as $\gamma\to m$ or $\gamma\to m+1$. Thus there are higher order phase transitions at each integral value $\gamma=2, 3, 4, ...$

In the case $\gamma=2$ we have $p(t)\sim(\zeta(2)\log\zeta(2)\,\xi/\log(1/\xi))$.

Note that depending on the class H_γ to which the potential g belongs, we are able to specify the order of lack of differentiability of $p(t)$.

Now we want to compute the moment generating function associated with our model. Define

$$p_n=\sum_{\sigma^n(\mathbf{x})=\mathbf{x}}e^{tS_n g(\mathbf{x})}\qquad(n=1, 2, ...),$$

where the summation is over all periodic $\mathbf{x}$ of (not necessarily minimal) period n and $S_n g(\mathbf{x})$ denotes $\sum_{i=0}^{n-1}g(\sigma(\mathbf{x}))$. Then we prove

PROPOSITION. $\lim_{n\to\infty}(1/n)\log p_n(t)=p(t)$.

To prove this, we compute the generating function $\sum_{n=1}^{\infty}p_n(t)\,z^n$ and see where its poles lie. The sum defining $p_n(t)$ has 2^n terms, corresponding to the possible choices of the first n of the a_i. If $a_0=0$, then we can write these as

$$a_0, ..., a_{n-1}=\underbrace{0\cdots0}_{m_0\geqslant 1}\ \underbrace{1\cdots1}_{n_1\geqslant 1}\ \underbrace{0\cdots0}_{m_1\geqslant 1}\cdots\underbrace{1\cdots1}_{n_r\geqslant 1}\ \underbrace{0\cdots0,}_{m_r\geqslant 0}$$

where r is the number of blocks of 1's ($r=0$ corresponds to $\mathbf{x}=(000\,...)$). In this case there are $m_0+\cdots+m_r=m$ values of $i\in\{0, 1, ..., n-1\}$ for which $\sigma^i(\mathbf{x})$ is in state 0, while during the remaining r blocks of length n_v the state of $\sigma^i(\mathbf{X})$ is successively $1, 2, ..., n_v$ $(v = 1, ..., r)$. Hence

$$S_n g(\mathbf{x}) = (m_0 + \cdots + m_r)\, a_0 + \sum_{v=1}^{r} (a_1 + \cdots + a_{n_v})$$

$$= -(m_0 + \cdots + m_r) \log(\zeta(\gamma)) - \gamma \sum_{v=1}^{r} \log(n_v + 1).$$

The contribution to $p_n(t)$ of this $\mathbf{x}$ is thus

$$\zeta(\gamma)^{-(m_0 + \cdots + m_r)t} (n_1 + 1)^{-\gamma t} \cdots (n_r + 1)^{-\gamma t}.$$

Similarly, if $a_1 = 1$ then we can write $a_0, ..., a_{n-1}$ as

$$\underbrace{1 \cdots 1}_{n_0 \geq 1}\ \underbrace{0 \cdots 0}_{m_1 \geq 1}\ \underbrace{1 \cdots 1}_{n_1 \geq 1} \cdots \underbrace{0 \cdots 0}_{m_r \geq 1}\ \underbrace{1 \cdots 1}_{n_r \geq 0}.$$

If $r = 0$, then $\mathbf{x} = (111 \ldots)$ and $S_n g(\mathbf{X}) = 0$. If $r > 0$, then the period of $\mathbf{x}$ consists of r blocks of 0's of total length $m_1 + \cdots + m_r$ and r blocks of 1's of lengths $n_1, n_2, ..., n_{r-1}$, and $n_0 + n_r$, so the contribution of $\mathbf{x}$ to $p_n(t)$ is

$$\zeta(\gamma)^{-(m_1 + \cdots + m_r)t} (n_1 + 1)^{-\gamma t} \cdots (n_{r-1} + 1)^{-\gamma t} (n_0 + n_{r-1} + 1)^{-\gamma t}.$$

This proves the formula

$$p_n(t) = \sum_{r \geq 0} \sum_{\substack{m_0, n_1, ..., n_r \geq 1 \\ m_r \geq 0 \\ m_0 + n_1 + \cdots + m_r = n}} \zeta(\gamma)^{-(m_0 + \cdots + m_r)t} (n_1 + 1)^{-\gamma t} \cdots (n_r + 1)^{-\gamma t}$$

$$+ \sum_{r \geq 1} \sum_{\substack{n_0, m_1, ..., m_r \geq 1 \\ n_r \geq 0 \\ n_0 + m_1 + \cdots + n_r = n}} \zeta(\gamma)^{-(m_1 + \cdots + m_r)t} (n_1 + 1)^{-\gamma t}$$

$$\cdots (n_{r-1} + 1)^{-\gamma t} (n_0 + n_{r-1} + 1)^{-\gamma t}.$$

Introducing the generating function simplifies this:

$$\sum_{n=1}^{\infty} p_n(t) z^n = \sum_{r=0}^{\infty} \left(\sum_{m=0}^{\infty} \zeta(\gamma)^{-mt} z^m \right)$$

$$\times \left(\sum_{m=1}^{\infty} \zeta(\gamma)^{-mt} z^m \right)^r \left(\sum_{n=1}^{\infty} (n+1)^{-\gamma t} z^n \right)^r - 1$$

$$+ \frac{z}{1-z} + \sum_{r=1}^{\infty} \left(\sum_{m=0}^{\infty} \zeta(\gamma)^{-mt} z^m \right)^r$$

$$\times \left(\sum_{n=1}^{\infty} (n+1)^{-\gamma t} z^n \right)^{r-1} \left(\sum_{n=1}^{\infty} n(n+1)^{-\gamma t} z^n \right)$$

(in the last term the extra factor n arises because each $n \geqslant 1$ can be written as $n_0 + n_r$ with $n \geqslant 1$, $n_0 \geqslant 0$ in exactly n different ways)

$$= \sum_{r=0}^{\infty} \left(\frac{1}{1-\zeta(\gamma)^{-t} z}\right)\left(\frac{\zeta(\gamma)^{-t} z}{1-\zeta(\gamma)^{-t} z}\right)^r \left(\sum_{n=1}^{\infty} \frac{z^n}{(n+1)^{\gamma t}}\right)^r - 1$$

$$+ \frac{z}{1-z} + \sum_{r=1}^{\infty} \left(\frac{\zeta(\gamma)^{-t} z}{1-\zeta(\gamma)^{-t} z}\right)^r \left(\sum_{n=1} \frac{z^n}{(n+1)^{\gamma t}}\right)^{r-1} \left(\sum_{n=1}^{\infty} \frac{n z^n}{(n+1)^{\gamma t}}\right)$$

$$= \frac{z}{1-z} + \frac{\zeta(\gamma)^{-t} \sum_{n=1}^{\infty} (z^n/n^{\gamma t-1})}{1-\zeta(\gamma)^{-1} \sum_{n=1}^{\infty} (z^n/n^{\gamma t})}.$$

For $t \geqslant 1$, we have $\sum_{n=1}^{\infty} (z^n/n^{\gamma t}) < \sum (1/n^{\gamma t}) = \zeta(\gamma t) \leqslant \zeta(\gamma)^t$ for all $z < 1$, so the power series $\sum p_n(t) z^n$ has its first pole at $z = 1$ and $\lim_{n \to \infty} p_n(t)^{-1/n} = 1$, in accordance with the theorem. For $t < 1$, the denominator of the second term vanishes at $z = e^{-p(t)}$, so the radius of convergence $\lim_{n \to \infty} p_n(t)^{-1/n}$ of $\sum p_n(t) z^n$ is $e^{-p(t)}$. The theorem follows.

From the above considerations is easy to see that the Ruelle zeta function (see the Introduction) is given by $\tilde{\zeta}_g(z) = (1-z)^{-1} (1 - \zeta(\gamma)^{-t} \sum_{m=1}^{\infty} (z^n/n^{\gamma t}))^{-1}$.

2. The Dynamic Zeta Function Associated with the Potential g and the Distribution of Periodic Orbits

The dynamic zeta function associated with the potential g is by definition

$$\zeta_g(t) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{\sigma^n(\mathbf{x}) = \mathbf{x}} e^{-t S_n g(\mathbf{x})} \right) 2^{-tn},$$

where $S_n g(\mathbf{x}) = \sum_{j=0}^{n-1} g(\sigma^j(\mathbf{x}))$.

The reason for the factor 2^{-tn} above is that the entropy of the shift is $\log 2$.

It is well known that when g is in F_θ (see the definition in Section 1), then $\zeta_g'(t)/\zeta_g(t)$ has a simple pole at $t = e^{P(g)}$ (see [26, 27]). This fact is related to differentiability of pressure [32]. In this case, a result analogous to the "distribution of prime number theorem" was obtained by Parry [24]. Several nice theorems related to this result are also known [15, 23, 26]. We refer the reader to a forthcoming book by W. Parry and M. Pollicot for an extensive exposition about the subject [26]. The main ingredient for obtaining such a "distribution of periodic orbits" type of result is the Tauberian theorem of Ikehara and Wiener and its generalizations [26, 40].

In the considerations of our paper, the map g is not in F_θ, and, in fact, $p(t)$ is not differentiable. As we know, for the type of singularity of $p(t)$ for

t close to 1, we will be able to apply a Tauberian theorem and obtain a "distribution of periodic orbits theorem."

We indicate now how to obtain such a result. The reader can check that in the considerations of Section 1, about the functional equation

$$\zeta(\gamma)^t = \sum_{n=1}^{\infty} \frac{e^{-np(t)}}{n^{\gamma t}},$$

we can consider $t = x + yi$ a complex vale in an angle sector centered in 1 intersecting the semi-axis $x < 1$. This allows one to take a branch of log and to consider derivatives. In this way, the estimates of the theorem in Section 1 are also true for t complex in the sector (see Postnikov [28, Sect. 3, Theorem 3, p. 11]).

We showed in the last section that $p(t) = \lim_{n\to\infty} (1/n) \log \sum_{\sigma^n(\mathbf{x}) = \mathbf{x}} e^{-tS_b g(\mathbf{x})}$.

From canonical considerations [28], for $t \sim 1$, we can therefore derive that $\zeta_g'(t)/\zeta_g(t)$ has the same behaviour as

$$\frac{(d/dt)(\exp \sum_{n=1}^{\infty} (1/n)\, e^{np(t)}\, 2^{-tn})}{\exp \sum_{n=1}^{\infty} (1/n)\, e^{np(t)}\, 2^{-tn}}$$

$$= \frac{(d/dt) \exp(\log(1 - e^{p(t)} 2^{-t}))}{\exp(\log(1 - e^{p(t)}\, 2^{-t}))} = \frac{(d/dt)(1 - e^{p(t)}\, 2^{-t})}{1 - e^{p(t)}\, 2^{-t}}.$$

(a) Let us now consider $2 < \gamma$. As we had seen before, $p(t) = h(\mu)(1-t) + A(1-t)^{\gamma-1} +$ high order terms. Therefore, $\zeta_g(t) \sim 1 - e^{h(\mu)(1-t) + A(1-t)^{\gamma-1}}$, and consequently as $\gamma > 2$

$$\frac{(d/dt)(1 - e^{p(t)}\, 2^{-t})}{1 - e^{p(t)}\, 2^{-t}}$$

$$\sim \frac{(-h(\mu) - A(\gamma-1)(1-t)^{\gamma-2})\, e^{h(\mu)(1-t) + A(1-t)^{\gamma-1}}}{h(\mu)(1-t) + A(1-t)^{\gamma-1} + \text{high order terms}}$$

$$\sim -\frac{1}{1-t}.$$

Now following [24, p. 46, (1.1); 26] we have that

$$\zeta_g(t) = \exp \sum_{n=1}^{\infty} \frac{1}{n} 2^{-tn} \sum_{\sigma^n(\mathbf{x}) = \mathbf{x}} e^{tS_n g(\mathbf{x})}$$

$$= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{\sigma^n(\mathbf{x}) = \mathbf{x} \\ h \text{ least per.}}} \sum_{k=1}^{\infty} \frac{1}{k} e^{-ktS_n g(\mathbf{x})}\, 2^{-tnk}$$

$$= \exp\left(-\sum_{n} \sum_{\substack{k=1 \\ J \text{ of per. } n}}^{\infty} \tfrac{1}{k} e^{ktS_n g(J)}\, 2^{-ntk}\right),$$

where J denotes a periodic orbit of minimum period n and $\lambda(J) = S_n\, g(J)$.

Now define $N(J)=e^{-\lambda(J)}2^n$ (sometimes called the norm of the periodic orbit J). Therefore, we can conclude that

$$\frac{\zeta_g'(t)}{\zeta_g(t)}=\sum_{k=1}^{\infty}\sum_J \log N(J)\,N(J)^{-tk}=-\frac{1}{1-t}+\text{bounded}.$$

Now defining

$$S(x)=\sum_{N(J)^n<x}\log N(J)=\sum_{N(J)<x} n\log N(J)$$

(where $n=[\log x/\log N(J)]$, i.e., the largest integer such that $N(J)^n\leqslant x$) we see that

$$\frac{\zeta_g'(t)}{\zeta_g(t)}=-\int_1^{\infty} x^{-t}\,dS(x)=-\frac{1}{1-t}+\text{bounded}.$$

Using classical results of Tauberian type [24, 26, 28], we obtain $S(x)\sim x$ for $x\sim\infty$. As in [26], we therefore conclude that if $\pi(x)=\sum_{N(J)\leqslant x}1$, then for $2<\gamma$ and $x\sim\infty$ we have $\pi(x)\sim x/\log x$.

(b) Now we consider $1<\gamma<2$. In this case $p(t)\approx B(1-t)^{1/(\gamma-1)}$. Therefore, $\zeta_g(t)\sim 1-e^{p(t)}=1-e^{(1-t)^{1/(\gamma-1)}}$, and consequently we have

$$\frac{\zeta_g'(t)}{\zeta_g(t)}\approx\frac{B(1-t)^{1/(\gamma-1)-1}\,e^{(1-t)^{1/(\gamma-1)}}}{(1-t)^{1/(\gamma-1)}+(1/2!)(1-t)^{2/(\gamma-1)}+\cdots}\approx B\frac{1}{(1-t)}.$$

Again using the reasoning of (a), we apply the Tauberian theorem to obtain, for $x\sim\infty$, that $S(x)\sim x$ and $\pi(x)\sim x/\log x$.

In conclusion to the above considerations we prove the following result:

THEOREM 2. *Consider $\pi(x)$ as $\sum_{N(J)\leqslant x}1$, where J are orbits with minimal period n and $N(J)=e^{-S_n(g(J))}2^n$. Then we have $\Pi(x)\sim x/\log x$.*

3. THE PRESSURE ASSOCIATED WITH THE MANNEVILLE–POMEAU MAP

The Manneville–Pomeau map f is a map from the interval $[0,1]$ in itself given by

$$f(x)=x+x^{1+s}\qquad(\text{mod }1),$$

where s is a positive constant.

In this section we are interested in analyzing the variational problem

$$P(t) = \sup_{\tilde{v} \in M(f)} \left\{ h(\tilde{v}) - t \int \log |f'(x)| \, d\tilde{v}(x) \right\},$$

where $M(f)$ denotes the set of f-invariant probabilities.

The dynamics of f is quite simple (see Fig. 2). In fact, f is conjugated with the shift in two symbols. The big trouble in analyzing such a map is related to the fact that f is not uniformly expanding (or sometimes called hyperbolic), because of the fixed indifferent point 0.

The important point in our considerations is that we have an analytical expression for $f'(x)$ for x close to 0. This expression is given by

$$f'(x) = 1 + (1+s)x^s, \qquad \text{for } x \text{ around } 0.$$

M. Thaler [34], using the above fact, shows the following result: there exists a density $h(x)$, $x \in (0, 1)$, such that $h(x)\,dx$ is invariant for f and there exist also constants $C_1, C_2 > 0$ such that $C_1 x^{-s} \geqslant h(x) \leqslant C_2 x^{-s}$.

The measure $h(x)\,dx$ is ergodic [34, 35] and is the Bowen–Ruelle–Sinai measure of our system.

In the case $s < 1$ we have that $h(x)\,dx$ is a probability, and in the case $s \geqslant 1$ we have that $h(x)\,dx$ is an invariant infinite measure.

In terms of the notation of Section 1 concerning the Ruelle–Perron–Frobenius operator, we should view $h(x)\,dx$ as $d\mu$ and dx as dv. Note that neither of these two measures is the maximal entropy measure.

From [35], the basic properties of the Ruelle–Perron–Frobenius operator that we need here can be derived from more general results.

We know from [21] that for any invariant measure $\tilde{v}$ the relation

$$\mathrm{HD}(\tilde{v}) = \frac{h(\tilde{v})}{\int \log |f'(x)| \, d\tilde{v}(x)}$$

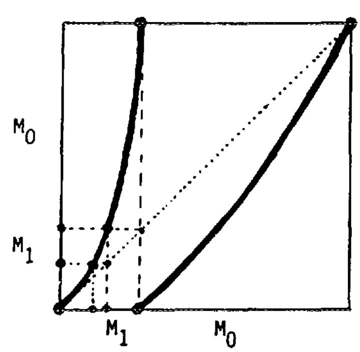

FIGURE 2

is true. Here $\mathrm{HD}(\tilde{v})$, the Hausdorff dimension of the measure $\tilde{v}$, is formally defined as $\inf\{\mathrm{HD}(A) \mid \tilde{v}(A)=1\}$. Therefore, $\mathrm{HD}(\tilde{v}) \leqslant 1$ for any $\tilde{v} \in M(f)$.

From the above fact, it follows easily that $P(t)=0$ for $t=1$, and $P(t) \leqslant 0$ for $t<1$.

As $f'(0)=1$, it also follows that $P(t)=0$ for $t \geqslant 0$ and the Dirac delta is zero in the equilibrium state for $P(t)$ in this case.

Now we show the following results:

THEOREM 3. *Suppose f from $[0,1]$ to itself is given by $f(x)=x+x^{1+s}$ $(\mathrm{mod}\ 1)$ with $s>0$. Then the pressure*

$$P(t)=\sup_{\tilde{v} \in M(f)}\left\{h(\tilde{v})-t \int \log |f'(x)|\, d\tilde{v}(x)\right\}$$

is zero for $t \geqslant 1$ and for $t<1$, we have the above expression for t close to zero:

$$P(t) \approx \begin{cases} A(1-t)+B(1-t)^{1/s}, & \text{for } s<1, s>0.5 \\ C(1-t)^{s}, & \text{for } s \geqslant 1. \end{cases}$$

Proof. We use the results previously obtained for the shift σ and potentials g in Section 1.

Consider $a \in (0,1)$ such that $1=f(a)=a+a^{1+s}$. Denote $x_0=a$, $x_1=$ the solution of $f(x_1)=x_0$ such that $x_1<x_0$, and inductively x_n the solution of $f(x_n)=x_{n-1}$, such that $x_n<x_{n-1}$. In this way we obtain a partition of $[0,1]$ in intervals of the form

$$\cdots(x_3, x_2], (x_2, x_1], (x_1, x_0](x_0, 1].$$

In order to stress the analogy with Section 2 (and the conjugation of f with the shift σ), we denote $(x_n, x_{n-1}]$ by M_n for $n \in \mathbb{N}$ and $M_0=(x_0, 1]$. Note that $f^n|_{M_n}$ is a diffeomorphism over M_0.

It follows from Proposition 2.4 in [35] that

$$(x_{n-1}-x_n) \sim \sup_{x \in M_n}\{f^{n'}(x)\} \sim \inf_{x \in M_n}\{f^{n'}(x)\}$$

(that is, all are of the same order).

Recall that there exists a natural change of coordinates for $[0,1]$ to Σ such that f is conjugated with σ. We can transfer results about pressure from one setting to another. In this way we make no distinction between a set in Σ and its correspondent under a change of coordinates on $[0,1]$. In this sense, the sets M_n, $n \in \mathbb{N}$, from Section 1 and those mentioned above are the same for us.

Consider the two variational problems

$$P_1(t) = \sup_{\tilde{v} \in M(\sigma)} \left\{ h(\tilde{v}) - t \int g_1(x)\, d\tilde{v}(x) \right\}$$

and

$$P_2(t) = \sup_{\tilde{v} \in M(\sigma)} \left\{ h(\tilde{v}) - t \int g_2(x)\, d\tilde{v}(x) \right\},$$

where for $x \in M_n$, $n \in \mathbb{N}$,

$$g_1(x) = \sup_{y \in M_n} \{\log |f'(y)|\} = a_n^1$$

and

$$g_2(x) = \inf_{y \in M_n} \{\log |f'(y)|\} = a_n^2.$$

It follows from the definition of $P(t)$ in the enunciation of Theorem 3, above, and Theorem 9.7(ii) in [36] that

$$P_2(t) \leqslant P(t) \leqslant P_1(t).$$

Note that

$$\log |f'(x_n)| = \inf_{y \in M_n} \{\log |f'(y)|\} = a_n^1$$

$$\sup_{y \in M_{n+1}} \{\log f'(y)\} = a_{n+1}^2$$

and that

$$f^{n'}(x_n) \sim (x_n - x_{n-1}) \qquad \text{(see [35])}.$$

From the last expression we can apply the results of F. Hofbauer mentioned at the beginning of Section 1.

Using the notation of Section 1 we denote by μ_1, v_1 and μ_k^1, v_k^1, $k \in \mathbb{N}$, the measures and real values obtained by the Ruelle–Perron–Frobenius theorem for g given by the a_n^1, $n \in \mathbb{N}$ (see [13]).

In the same way we denote by μ_2, v_2 and μ_k^2, v_k^2, $k \in \mathbb{N}$, the measures and real values obtained from the Ruelle–Perron–Frobenius theorem for g given by the a_n^2, $n \in \mathbb{N}$ (see [13]).

The measures μ, v obtained from the Ruelle–Perron–Frobenius theorem for $g(x) = -\log |f'(x))|$, $x \in [0, 1]$, are squeezed as in a sandwich between μ_1 and μ_2, respectively, and v_1 and v_2, respectively.

From the same considerations of F. Hofbauer on page 226 of [13], we obtained that $\nu_k = \nu(M_k)$ and $\mu_k = \mu(M_k)$ are, respectively, of the same order as ν_k^1, ν_k^2 and μ_k^1, μ_k^2. Therefore, μ_k has the same order as $\sum_{i=k}^{\infty} (1/i)$, and this summation has the same order as $\sum_{i=k}^{\infty} |f^{i'}(x_i)| \sim \sum_{i=k}^{\infty} (x_{i-1} - x_i) = x_{k-1}$.

The value $\mu_k = \mu(M_k) = \int_{x_k}^{x_{k-1}} h(x)\, dx \sim x_{k-1}^{-s+1} - x_k^{-s+1}$. As x_k/x_{k-1} goes to one as k goes to ∞ (because $f'(0)=1$) we have $x_{k-1}^{-s} - x_k^{-s} \sim C$, where C is a constant. From this it follows easily that (see [1, p. 250]) $x_k \sim (Ck)^{-1/s}$. Therefore,

$$x_{k+1} - x_k = x_k + x_k^{1+s} - x_k$$

is of order $k^{-((1+s)/s)} = k^{-\gamma}$ with $\gamma = 1 + 1/s$. Remember that $x_{k+1} - x_k$ is of the same order as $e^{a_0^1 + a_2^1 + \cdots a_k^1}$.

From Remark 1 in Section 1 we conclude that $P_1(t)$ and $P_2(t)$ have the behaviour given by Theorem 1 and depend on $\gamma = 1 + 1/s$.

As $P_1(t) \leqslant P(t) \leqslant P_2(t)$, the same conclusions apply for $P(t)$. Note that $s > 1$ if and only if $1 < \gamma < 2$.

This is the end of the proof of Theorem 3.

4. Random Walks in the Lattice $\mathbb{N}$

In this section we give another proof of Theorem 1 concerning the asymptotic behaviour of $p(t)$ for values of t close to 1. We present a mathematically rigorous proof of the reasoning considered in Gaspard and Wang [12] and Wang [37, 38]. We do not obtain in this way the exact values of the constants in the asymptotic expansion of $p(t)$, as in the last section, but following the beautiful ideas of X.-J. Wang [37, 38], we show a relation of the model considered above with random walks in lattices.

Consider the random walk on $\mathbb{N}$ by the transition probabilities

$$p_{0n} = \zeta(\gamma)^{-1} (n+1)^{-\gamma}, \qquad n \in \mathbb{N}$$

and the transition matrix

$$A = \begin{pmatrix} p_{00} & 1 & 0 & 0 & \cdots \\ p_{01} & 0 & 1 & 0 & \cdots \\ p_{02} & 0 & 0 & 1 & \cdots \\ p_{03} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The invariant vector for A is the infinite sequence $(\mu(M_0), \mu(M_1), \mu(M_2), \ldots)$, as we show later.

Following F. Hofbauer, consider for γ fixed and $t \in \mathbb{R}$ the values

$$v_{k,t} = \lambda_t^{-k-1} \zeta(\gamma)^{-t} (k+1)^{-\gamma t} = v_t(M_k), \qquad k \in \mathbb{N}$$

and

$$v_{0,t} = \lambda_t^{-1} \zeta(\gamma)^{-t} = v_t(M_0),$$

where

$$\log \lambda_t = p(t) = \sup_{\hat{v} \in M(\sigma)} \left\{ h(\hat{v}) - t \int g_\gamma(z)\, d\hat{v}(z) \right\}.$$

We also use the notation below:

For $t = 1$, $v_t = v_1 = v$ and the equilibrium state $\mu_t = \mu_1 = \mu$. Consider also $\mu = (\sum_{j=1}^{\infty} j v_{j-1,1})^{-1}$. In the case $1 < \gamma \leqslant 2$, we have $\mu = 0$ and for $2 < \gamma < 3$, we have $\mu = \zeta(\gamma)/\zeta(\gamma - 1) > 0$.

(a) First we consider the case $2 < \gamma < 3$. For $\varphi = t g_\gamma$, $2 < \gamma < 3$, the eigenfunction h_t mentioned in Section 1, the eigenvalue of the Ruelle–Perron–Frobenius operator (that is, $\mathscr{L}_{t\varphi} h = \lambda_t h_t$) satisfies $h_t(\mathbf{x}) = \mu v_{k,t}^{-1} \sum_{i=k}^{\infty} v_{i,t}$ for $\mathbf{x} \in M_k$, and $h_t(1111\ldots) = \mu \sum_{i=0}^{\infty} \lambda_t^{-1}$. We denote h_1 by h. Note that for $t = 1$, $h(111\ldots) = \infty$. Note also that in any case (that is, $t \leqslant 1$), $h(\mathbf{x}) \geqslant \mu > 0$ (see [13, p. 229]) for $x \in \Sigma - \{(1, 1, 1, \ldots)\}$. We also have that for $\mathbf{x} \in M_k$, $\mu_k = \mu(M_k) = h(\mathbf{x}) v_{k,1}$. Denote $v_{k,1} = v_k$; therefore,

$$\mu_k = \mu \sum_{i=k}^{\infty} v_i = \mu \sum_{i=k}^{\infty} e^{a_0} (i+1)^{-\gamma} = \mu \sum_{i=k}^{\infty} p_{0i},$$

and this shows that (μ_k) is invariant for A, that is, $A((\mu_k)) = (\mu_k)$ (see [2, p. 105]).

Finally, from elementary calculus μ_k grows like $k^{1-\gamma}$. This fact will be important later.

We also know that $\mu(M_k) = \mu_k$, where μ is the eigenmeasure of $\mathscr{L}_\varphi^*$ when $\varphi = g_\gamma$, $2 < \gamma < 3$. This fact creates a relation of μ and the random walk. The reason to consider the random walk is related to the future use of Feller's result concerning recurrent events and its relation to the stable law of Levy [10].

We explain now the reason why it is natural to consider the random walk with such transition probabilities. We want to associate with each element $(\mathbf{x})$ in the lattice the infinite sequence (random walk path) $(k_0, k_1, k_2, k_3, \ldots)$ such that $\sigma^i(\mathbf{x}) \in M_{k_i}$. We do not want to consider $\mathbf{x} = (1, 1, \ldots, 1 \ldots)$ or its inverse images. Therefore, this identification is considered in $\tilde{\Sigma} = \Sigma - \{\sigma^{-n}\{(1, 1, \ldots)\} \mid n \in \mathbb{N}\}$. Therefore, an element $\mathbf{x} \in \tilde{\Sigma}$ can be seen as a random walk path.

We explain now why we define p_{0_n} in the above way.

Let us compute the transition probabilities in a certain way that is closely related to μ. If $\mathbf{x} \in M_0$, then the condition probability of $\sigma(\mathbf{x}) \in M_0$ is associated with the set $M_{00} = \{(0, 0, y_2, ...) \mid y_i \in \{0, 1\},\ i \geqslant 2\}$. In fact, $\mu(M_{00})/\mu(M_0)$ is the conditional probability of $\sigma(\mathbf{x})$ seen in M_{00} given that $\mathbf{x} \in M_0$.

As $\mu(M_0) = \mu(M_{00}) + \mu(M_1)$ (because μ is invariant), then

$$1 = \frac{\mu(M_{00})}{\mu(M_0)} + \frac{\mu(M_1)}{\mu(M_0)} = p_{00} + \frac{\mu(M_1)}{\mu(M_0)}$$

or, in other terms,

$$\mu(M_0) = \mu(M_1) + p_{00}\, \mu(M_0).$$

Now, if $\mathbf{x} \in M_0$, the conditional probability that $\sigma(\mathbf{x}) \in M_1$ is given by $p_{01} = \mu(M_{01})/\mu(M_0)$, where $M_{01} = \{(0, 1, y_2, y_3, ...) \mid y_i \in \{0, 1\},\ i \geqslant 2\}$. As μ is invariant $\mu(M_1) = \mu(M_{01}) + \mu(M_2)$; therefore,

$$\frac{\mu(M_1)}{\mu(M_0)} = p_{01} + \frac{\mu(M_2)}{\mu(M_0)}$$

and

$$\mu(M_1) = p_{01}\, \mu(M_0) + \mu(M_2).$$

By induction we have

$$\mu(M_{n-1}) = p_{0n-1}\, \mu(M_0) + \mu(M_n) \qquad \text{for all} \quad \mu \in \mathbb{N},$$

and

$$\mu(M_{n-1}) = \frac{\mu(M_{\underbrace{011\cdots10}_{n+2}})}{\mu(M_0)},$$

where

$$M_{\underbrace{011\cdots10}_{n+2}} = \{(\underbrace{0, 1, 1, ..., 1, 0,}_{n+2}\ y_{n+3}, y_{n+4}, ... \mid y_i \in \{0, 1\}, i \geqslant n+3\}.$$

As $\sigma(M_n) = M_{n-1}$, the conditional probability of $\sigma(\mathbf{x}) \in M_{n-1}$ given that $\mathbf{x} \in M_n$ is one (that is, $p_{nn-1} = 1$ for all $n \geqslant 1$). All other p_{ij} different from those considered above are zero. Therefore, in this way we have another proof that the infinite vector $z = (\mu(M_0), \mu(M_1), \mu(M_2), ...)$ is invariant when we apply the infinite matrix A, that is, $A(z) = z$.

From the above considerations, one can see that it is quite reasonable to consider such a random walk with the transition probabilities assigned above.

A simple computation shows that $\mu\{(111, \ldots)\} = v\{(111, \ldots)\} = 0$; therefore, $\sum_{k=0}^{\infty} \mu_k = 1$. In other words, $\mu(\Sigma - \tilde{\Sigma}) = 0$.

Now we would like to make clear an important point (see [2, p. 104]). Given (μ_k), the invariant vector for A (recall that $\sum_{k=0}^{\infty} \mu_k = 1$ and p_{st} are the transition elements of A), we can obtain a Markov shift in the following way: consider

$$X = \{(x_0, x_1, x_2, \ldots, x_n \ldots) \mid x_i \in \mathbb{N},\ i \in \mathbb{N}\}$$

and define

$$\begin{aligned} &\mu_A([x_n = s_n, x_{n+1} = s_{n+1}, \ldots, x_{n+k} = s_{n+k}]) \\ &\quad = \mu_{s_n} p_{s_n s_{n+1}} \cdots p_{s_{n+k-1} s_{n+k}} \\ &\qquad\qquad \text{for} \quad k \geqslant 1,\ s_n, s_{n+1}, \ldots, s_{n+k} \in \mathbb{N}. \end{aligned}$$

We can identify points in the lattice $\tilde{\Sigma} = \Sigma - \{\sigma^{-n}(1, 1, 1, \ldots) \mid n \in \mathbb{N}\}$ with paths in the random walk mentioned above.

The measure μ_A defined above is the same measure μ in the identified spaces (respectively, random walk and $\tilde{\Sigma}$). Therefore, we can identify the random variable N_n (in the random walk setting), the number of passages to 0 during n units of time with the random variable (in Σ or $\tilde{\Sigma}$), as

$$N_n(\mathbf{x}) = \sum_{j=0}^{n-1} I_{M_0}(\sigma^j(\mathbf{x})),$$

where I_{M_0} is the indicator of the set M_0.

The probability that p_n^{00} returns at time n for the first time to zero beginning in zero is given by

$$\mu(M_{\underbrace{011\cdots10}_{n+2}}).$$

These values can be obtained by the fact that $\mu \in M(\sigma)$. For instance, $\mu(M_{00}) + \mu(M_1) = \mu(M_0)$; therefore, $\mu(M_{00}) = \mu(1 - \sum_{n=1}^{\infty} (n+1)^{-\gamma} \zeta(\gamma)^{-1})$. As for $(M_{010}) + \mu(M_2) = \mu(M_1)$, then $\mu(M_{010}) = \mu 2^{-\gamma} \zeta(\gamma)^{-1}$.

Inductively

$$\mu(M_{\underbrace{011\cdots10}_{n+2}}) = \mu(n+1)^{-\gamma} \zeta(\gamma)^{-1},$$

and therefore, p_n^{00} decreases like $n^{-\gamma}$.

For each $n \in \mathbb{N}$, we denote

$$\underbrace{M_{011\cdots 10}}_{n+1}$$

by Q_n. Note that $\sigma(M_n) = \sigma(Q_n) = M_{n+1}$. In particular $M_{00} = Q_1$ and $\sigma(M_{00}) = \sigma(Q_1) = \sigma(M_1) = M_0$.

Define $F(x) = \sum_{n=0}^{[x]} p_n^{00}$ as the distribution function of first return in the random walk. Note that we use the identification of μ_A and μ as above.

From the above, we know that $1 - F(x)$ decreases like $x^{1-\gamma}$.

Following Wang [37, 38], we use $1 - F(x) \sim x^{-\gamma+1} = x^{-\alpha}$ and Feller [10, Theorem 7, p. 106] to conclude that N_n has a distribution $G_{1-\gamma}$ of Levy type, that is,

$$\Pr\left\{N_k \geqslant \frac{k}{M} - \frac{b_k}{M^{\gamma/(\gamma-1)}}x\right\} \to G_{1-\gamma}(x),$$

where b_k is such that $1 - F(b_k) \sim 1/k$, and M is the mean recurrence time

$$M = \sum_{n=1}^{\infty} np_n^{00}.$$

Remark 4. From the results obtained here we have the more precise estimate $1 - F(x) \approx x^{1-\gamma} \cdot \zeta(\gamma)^{-1}$ (see Remark 9).

We point out the importance of the identification considered above (lattice $\tilde{\Sigma}$ and random walk), because now we have a non-trivial result for μ on $\tilde{\Sigma}$ obtained by means of random walk considerations.

Consider now, for $q > 0$, $\phi(q) = \lim_{n\to\infty}(1/n) \log \int e^{-qN_n(w)}\, d\mu(w)$, the free energy associated with the random variable I_{M_0} [37]. Now we use an important technical lemma proved by Wang in Appendix C of [38]. The proof presented there is rigorously correct mathematically.

LEMMA 2. *Under the above definitions*

$$\phi(q) = \begin{cases} |q| + A\,|q|^{\alpha}\,(\Gamma(1-\alpha)/J^{\alpha+1}) & \text{if } q < 0 \\ 0 & \text{if } q > 0, \end{cases}$$

where $\gamma = \alpha + 1$, *and* A, J *are constants.*

We refer the reader to the proof in [38]. The main idea of the proof is to use the known fact that N_n has a distribution of Levi type, and properties of the Laplace transform.

Now we must show a delicate estimate relating $\phi(q)$ to $p(t)$, where

$(1-t)=q$. In fact, we show that for t close to 1 (and q close to zero), we have $p(t) \sim \phi(q)$.

In this part we were not able to follow the reasoning of [37, 38]. It is not clear to the author of this paper if the reasoning can be done in that way. We present a different reasoning to show the claim.

The main point in the proof below is that instead of finding universal constants C_1 and C_2 (depending only on γ) to bound the quotients

$$C_1 \leqslant \frac{\mu(\overline{x_0, x_1, ..., x_{n-1}})}{\exp(S_n(g(\mathbf{x})))} \leqslant C_2, \qquad \mathbf{x} \in \overline{x_0, x_1, ..., x_{n-1}}, n \in \mathbb{N},$$

where $\overline{x_0, x_1, ..., x_{n-1}}$ is the cylinder $\{(x_0, x_1, x_2, ..., x_{n-1}, z_m, x_{m+1}, ...) \mid z_i \in \{0, 1\}, i \geqslant m\}$, we consider bounds of the form $n^{-\Delta}$, $\Delta \in \mathbb{R}$. Therefore, instead of using $\lim_{n \to \infty} (1/n) \log C = 0$, where C is a constant, we use the fact that $\lim_{n \to \infty} (1/n) \log n^{-\Delta} = 0$, to obtain our result.

Remark 5. Note that the measure μ is not a homogeneous measure (see the theorem on page 230 of [13]); that is, there are no constants C_1 and C_2 independent of n such that the last inequalities occur.

The above claim is used in [19, Sect. 2] in an essential way.

We now begin the proof of the claim $p(t) \sim \phi(q)$, $(1-t)=q$.

For each $n \in \mathbb{N}$ we have 2^n cylinders of the form $\overline{x_0, x_1, ..., x_{n-1}}$. Denote the 2^{n-1} cylinder of form $\overline{1, x_1, x_2, ..., x_{n-1}}$ by C_n^i, $i \in \{1, 2, ..., 2^{n-1}\}$, and denote the 2^{n-1} cylinders of form $\overline{0, x_1, x_2, ..., x_{n-1}}$ by d_n^i, $i \in \{1, 2, ..., 2^{n-1}\}$.

We denote $C_n^1 = \overline{111 \cdots 11}$ and $d_n^1 = \overline{0111 \cdots 1}$. Note that $\bigcup_{i=1}^{2^{n-1}} d_n^1 = M_0$ and $\bigcup_{i=1}^{2^{n-1}} C_n^i = \Sigma - M_0$.

Denote by P_n the set of C_n^i and d_n^i, $i \in \{1, 2, ..., 2^{n-1}\}$. Therefore, P_n has cardinality 2^n.

We need the following lemma, which we prove later:

LEMMA 3. *Under the above definitions, we have that*

$$\begin{aligned} 0 &= \lim_{n \to \infty} \frac{1}{n} \log \inf_{\substack{C_n^i \in P_n \\ \mathbf{x} \in C_n^i \\ i > 1}} \frac{\mu(C_n^i)}{\exp S_n(g(\mathbf{x}))} \\ &= \lim_{n \to \infty} \frac{1}{n} \log \sup_{\substack{C_n^i \in P_n \\ \mathbf{x} \in C_n^i \\ i > 1}} \frac{\mu(C_n^i)}{\exp S_n(g(\mathbf{x}))} \end{aligned}$$

and also

$$0=\lim_{n\to\infty}\frac{1}{n}\log\inf_{\substack{d_n^i\in P_n\\ \mathbf{x}\in d_n^i\\ n>1}}\frac{\mu(d_n^i)}{\exp S_n(g(\mathbf{x}))}$$

$$=\lim_{n\to\infty}\frac{1}{n}\log\sup_{\substack{d_n^i\in P_n\\ \mathbf{x}\in d_n^i\\ n>1}}\frac{\mu(d_n^i)}{\exp S_n(g(\mathbf{x}))}.$$

Remark 6. Note the important restriction $i>1$ in the claim of the lemma above.

Let us suppose that Lemma 3 is proved. Note that for given $w_1, w_2\in C_n^i$ (or $w_1, w_2\in d_n^i$), $i\in\{1, 2, 3, ..., 2^{n-1}\}$, $e^{-qN_n(w_2)-qn}\leqslant e^{-qN_n(w_1)}\leqslant e^{-qN_n(w_2)+qn}$. Therefore, we have

$$-q+\lim_{n\to\infty}\frac{1}{n}\log\int e^{-qN_n(w)}\,d\mu(w)$$

$$\leqslant\lim_{n\to\infty}\frac{1}{n}\log\sum_{\substack{x\in p_n\\ (\mathbf{y})\in x\\ \sigma^n(\mathbf{y})=\mathbf{y}}}e^{-qN_n(\mathbf{y})}\mu(x)$$

$$\leqslant q+\lim_{n\to\infty}\frac{1}{n}\log\int e^{-qN_n(w)}\,d\mu(w).$$

As we consider q small, $\phi(q)$ has the same order as

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{\substack{x\in p_n\\ \mathbf{y}\in x\\ \sigma^n(\mathbf{y})=\mathbf{y}}}e^{-qN_n(\mathbf{y})}\mu(x).$$

For $\mathbf{y}\in C_n^1$ and $\sigma^n(\mathbf{y})=\mathbf{y}$, we have $e^{-qN_n(\mathbf{y})}\mu(C_n^1)=\mu(C_n^1)$ of order $n^{1-\gamma}$.

For $\mathbf{y}\in d_n^1$ and $\sigma^n(\mathbf{y})=\mathbf{y}$, we have also that $e^{-qN_n(\mathbf{y})}\mu(d_n^1)=e^{-q}\mu(d_n^1)$ is of order $n^{-\gamma}$.

We use the following notation: Σ' denotes summation over $x\in P_n$, $x\neq C_n^1$, $x\neq d_n^1$, $\mathbf{y}\in x$, $\sigma^n(\mathbf{y})=y$. Therefore, $\phi(q)$ has the same order as $\lim_{n\to\infty}(1/n)\log\Sigma' e^{-qN_n\mathbf{y}}\mu(x)$ (here we use $\phi(q)>0$, $q>0$).

Now from the lemma, we can substitute $\exp S_n\,g(\mathbf{y})$ for $\mu(x)$. Therefore, $\phi(q)$ has the same order as

$$\lim_{n\to\infty}\frac{1}{n}\log\Sigma' e^{-qN_n(\mathbf{y})+S_n(g(\mathbf{y}))}.$$

Now we must compare $-qN_n(\mathbf{y})+S_n g(\mathbf{y})$ with $tS_n g(\mathbf{y})$. Using the notation of Section 1, we have that the first summand above is equal to

$$-qm+\log[((n_1+1)\cdots(n_k+1))^{-\gamma}\zeta(\gamma)^{-m}].$$

where $m=N_n(y)$.

Since

$$\frac{|N_n(\mathbf{y})|}{|S_n g(\mathbf{y})|}=\frac{m}{\gamma\sum_{i=1}^k \log(n_i+1)+m\log\zeta(\gamma)} \leqslant \frac{m}{m\log\zeta(\gamma)}=\frac{1}{\log\zeta(\gamma)},$$

we conclude that the quotient $(-qN_n(\mathbf{y})+S_n g(\mathbf{y}))/tS_n g(\mathbf{y})$ is of order $(-q/t)E+1/t$, with q small, t close to one, and E constant. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\log\Sigma' e^{\{tS_n(g(\mathbf{y}))\}}$$

is of the same order as

$$\lim_{n\to\infty}\frac{1}{n}\log\Sigma' e^{-qN_n(\mathbf{y})+S_n(g(\mathbf{y}))}.$$

We showed before that the last expression has the same order as $\phi(q)$.

The value $e^{tS_n(g(\mathbf{y}))}$ for $\mathbf{y}\in C_n^1$ (respectively, $\mathbf{y}\in d_n^1$) has (by Remark 7, below, after the proof of Lemma 2) the same order as $\nu(C_n^1)$ (respectively, $\nu(d_n^1)$) and decreases like $n^{-\gamma}$; therefore, $\lim_{n\to\infty}(1/n)\log\Sigma' e^{tS_n}(g(\mathbf{y}))$ $((1-qE)/t)$ has the same order as $\lim_{n\to\infty}(1/n)\log\sum_{\sigma^n(\mathbf{y})=\mathbf{y}} e^{-tS_n(g(\mathbf{y}))}$.

Finally, we know (see [13, p. 225]) that

$$p(t)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{x\in P_n}\sup_{\mathbf{x}\in x}\{\exp tS_n(g(\mathbf{x}))\}.$$

By Lemma 3, for $\mathbf{x}\in x$, $y\in x$, (x is an atom P_n) $(\exp S_n(g(\mathbf{x})))/\nu(x)$ and $(\exp S_n(g(\mathbf{y})))/\nu(x)$ are of the same order (up to a factor n^{Δ} for $\Delta\in\mathbb{R}$). Therefore, $\exp S_n(g(\mathbf{x}))/\exp S_n(g(y))$ are also of the same order (up to a factor n^{Δ} for $\Delta\in\mathbb{R}$).

From this we conclude that (see also the proposition in Section 1)

$$p(t)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\sigma^n(\mathbf{y})=\mathbf{y}}\exp tS_n(g(\mathbf{y})).$$

From the considerations above, we conclude that $p(t)$ and $\phi(q)$ are of the same order. Therefore, using the lemma of Wang [38] we have that there exist constants A, B such that

$$p(t) \approx \begin{cases} A(1-t) + B(1-t)^{\gamma-1} & \text{for } t<1 \\ 0 & \text{for } t \geqslant 1. \end{cases}$$

In this way, we formalize the reasoning of Wang [37, 38].

Now we show the proof of Lemma 3.

Our proof is a variation of the argument used on page 231 of [13]. First note that even that h is not bounded above, h restricted to the cylinders C_n^i, $1<i\leqslant 2^{n-1}$, and d_n^i, $1<i<2^{n-1}$, has an upper bound of order $n^{\gamma}\cdot n^{1-\gamma}=n$, and therefore, as $\mu = hv$, we can analyze $v(C_n)$ instead of $\mu(C_n)$ for $n>1$.

Here we again use the fact that for $\Delta \in \mathbb{R}$, $\lim_{n\to 0}(1/n)\log n^{\Delta}=0$. Recall that $e^{S_n(g(w))}$ is of the form $[(n_1+1)(n_2+1)\cdots(n_k+1)]^{-\gamma}\zeta(\gamma)^{-m}$, where $m=m_1+m_2+\cdots+m_k$. Note that all values $n_1, ..., m_1, ..., k, m$ depend on w.

If $m_k=0$, then suppose $\sigma^{n-n_k}(w)\in M_l$ and therefore $\sigma^n(w)\in M_{l-n_k}$. In this case $e^{S_n(g_\gamma(w))}$ is equal to

$$\left[(n_1+1)(n_2+1)\cdots(n_{k-1}+1)\left(\frac{l+1}{l-n_k}\right)\right]^{-\gamma}\zeta(\gamma)^{-m}.$$

Now suppose $w\in\overline{x_0, x_1, ..., x_{n-1}}$ in our considerations is such that $m_k=0$. (In the case $m_k\neq 0$ the argument is simpler.) Using the above notation we have

$$\sigma^{n-n_k}(w)\in M_l \qquad \text{and} \qquad \sigma^n(w)\in M_{l-n_k}.$$

Note that l depends on w (see Remark 6). Following page 231 of [13] we have from the RPF condition

$$v[\overline{x_0, x_1, ..., x_{n-1}}] = \exp S_{n-n_k}(g(\mathbf{y}))\sum_{i=n_k}^{\infty} v_i,$$

where $\mathbf{y}\in\overline{x_0, x_1, ..., x_{n-1}}$. This fact follows from $\mathscr{L}_g^*(v)=v$. By another way

$$\begin{aligned} S_n(g(\mathbf{y})) &= S_{n-n_k}(g(\mathbf{y})) + a_l + \cdots + a_{l-n_k} \\ &= S_{n-n_k}(g(\mathbf{y})) + \log\left(\frac{l+1}{l-n_k}\right)^{-\gamma}. \end{aligned}$$

Consider now the quotient

$$\frac{\nu(\overline{x_0, x_1, ..., x_{n-1}})}{\exp(S_n(g(y)))} = \frac{\sum_{i=n_k}^{\infty} \nu_i}{((l+1)/(l-n_k))^{-\gamma}} \quad \text{for} \quad y \in \overline{x_0, x_1, ..., x_{n-1}}. \qquad (*)$$

For a fixed $n \in \mathbb{N}$, consider a fixed n_k such that $0 \leqslant n_k \leqslant n$, and $l > n_k$.

The above quotient is up to a constant of the form

$$\frac{n_k^{1-\gamma}}{((l+1)/(l-n_k))^{-\gamma}} = n_k^{1-\gamma} \left(\frac{l+1}{l-n_k}\right)^{\gamma}.$$

Note that $((l+1)+1)/((l+1)-n_k) < (l+1)/(l-n_k)$ and that $\lim_{l \to \infty}((l+1)/(l-n_k)) = 1$. Therefore, we have

$$n > n_k \geqslant \frac{n_k^{1-\gamma}}{(n_k)^{-\gamma}} > \frac{n_k^{1-\gamma}}{((l+1)/(l-n_k))^{-\gamma}} \geqslant n_k^{1-\gamma} > n^{1-\gamma}.$$

From the above inequalities, we conclude the proof of the lemma. Here we again use the fact that $\lim_{n \to \infty}(1/n) \log n^{\Delta} = 0$ for $\Delta \in \mathbb{R}$ fixed.

This handles the part $i > 1$.

Remark 7. Note that for $y \in C_1^n$, $\exp(S_n(g(y)))$ and $\nu(C_1^n)$ are of the same order up to some n^{Δ} (See $(*)$.)

From the remark the result is proved.

(b) Now we consider the case $1 < \gamma < 2$.

The major difference from case (a) is that the real value $\mu = [\sum_{j=1}^{\infty} j\nu_{j-1}]^{-1}$ is now equal to zero. Nevertheless, there exists a probability ν such that $\mathcal{L}_g^*(\nu) = \nu$ and, as before, $\nu_k = \nu(M_k) = e^{S_k}$. The eigenfunction h is given by $h(x) = \nu_k^{-1} \sum_{i=k}^{\infty} \nu_k$ (there is no normalizing factor μ).

The measure μ is defined by $\mu = h\nu$ and it is not a probability but an infinite measure. Therefore, in the transition value of the parameter $t = 1$, the only equilibrium state is $\delta_{111\ldots}$.

This fact can be seen as a second order transition (see [19]). Note that the reasoning used before for p_{0n}, p_n^{00}, and $\tilde{p}_n^{00}$ also applies. This follows from the fact that $\mu(M_k)$ is finite for each $k \in \mathbb{N}$. Therefore, we can derive the same results as before, but with the difference that now

$$\phi(q) = \begin{cases} |q|^{1/\alpha} B & \text{if } q < 0 \\ 0 & \text{if } q > 0, \end{cases}$$

where B is a constant.

The computation of the above fact is rigorously obtained in Wang [38]. As in (a) we finally conclude that

$$P(t)=\begin{cases} C(1-t)^{1/(\gamma-1)}, & t<1 \\ 0, & t\geqslant 1, \end{cases}$$

where C is a constant (the exact value of C was obtained in Section 1).

Following Wang [38] we can derive the following interesting results, explained below.

For each natural number $n\in\mathbb{N}$ consider the autocorrelation function on $C(n)$,

$$C(n)=\int I_{M_0}(\sigma^k(\mathbf{x}))\, I_{M_0}(\mathbf{x})\, d\mu(x),$$

and $S(w)$, the Fourier transform of the autocorrelation, that is,

$$S(w)=\sum_{n=0}^{\infty} C(n)\, e^{iwn}, \qquad w\in[-\pi,\pi).$$

From Theorem 10 of [10] we have

(a) for $1<\gamma<2$ and $n\sim\infty$, $\operatorname{var}(N_n)\sim n^{2(\gamma-1)}$, $E(N_n)\sim n^{\gamma-1}$, and

(b) for $2<\gamma<3$ and $n\sim\infty$, $\operatorname{var}(N_n)\sim n^{4-\gamma}$, $E(N_n)\approx n$.

It is well known that

$$\operatorname{var}(N_n)=\int_{-\pi}^{\pi}\left(\frac{\sin(wn/2)}{\sin(w/2)}\right)^2 S(w)\, dw.$$

Therefore, from the kind of singularity of $\operatorname{var}(N_n)$, we can obtain $S(w)\sim w^{3-2\gamma}$ for $w\sim 0$ if $1<\gamma<2$.

The final conclusion from $S(w)=\sum_{n=0}^{\infty} C(n)\, x^n$, $x=e^{iw}$, and the Tauberian theorem [10] is that

$$C(n)\sim n^{2-\gamma}, \qquad \text{when} \quad 2<\gamma<3.$$

In an analogous reasoning $C(n)\sim n^{-2(2-\gamma)}$, when $1<\gamma<2$.

Remark 8. Measures that are equilibrium states of functions g in the class F_θ have exponential decay of correlation [26]. In our case we do not have this property.

Remark 9. Note that when $1 < \gamma < 2$ and $n \sim \infty$, there follow from our results the more precise estimates

$$E(N_n) \approx \zeta(\gamma) \cdot \frac{\sin(\gamma - 1)\pi}{(\gamma - 1)\pi} n^{\gamma - 1}.$$

This follows from Remark 4 of this paper and Theorem 10 in [10].

Remark 10. The results concerning $S(w)$ are related to interesting considerations in [39] concerning $1/f$ and $1/f^{\alpha}$ noise.

Acknowledgments

I thank C. Berenstein and F. Olver for helpful discussions about asymptotics. I especially thank D. Zagier, who explained to me how to find the asymptotics of the free energy $p(t)$ and the moment generating function $\sum p_n x^n$.

References

1. J. Aaronson, Ergodic theory for inner functions on the upper half plane, *Ann. Inst. H. Poincaré Probab. Statist.* **14**, No. 3 (1978), 233–253.
2. J. Aaronson, Rational ergodicity and a metric invariant for Markov shifts, *Israel J. Math.* **27**, No. 2 (1977), 93–123.
3. T. M. Apostol, "Introduction to Analytic Number Theory," Springer-Verlag, New York/Berlin, 1976.
4. A. Ben-Mizrachi, I. Proccacia, N. Rosenberg, A. Schmidt, and H. G. Schuster, Real and apparent divergencies in low-frequency spectra of nonlinear dynamical systems, *Phys. Rev. A* **31**, No. 3 (1985), 1830–1840.
5. T. Bohr and M. Jensen, Order parameter, symmetry breaking, and phase transitions in the description of multifractals, *Phys. Rev. A.*, No. 10 (1987), 4904–4915.
6. R. S. Ellis, "Entropy, Large Deviation and Statistical Mechanics," Springer-Verlag, New York/Berlin.
7. M. Feigenbaun, M. Jensen, and I. Procaccia, Time ordering and the thermodynamics of strange sets: Theory and experimental tests, *Phys. Rev. Lett.* **57**, No. 13 (1986), 1503–1506.
8. M. Feigenbaun, I. Procaccia, and T. Tel, Scaling properties of multifractals as an eigenvalue problem, *Phys. Rev. A.* **39**, No. 10 (1989), 5359–5372.
9. B. Felderhof and M. Fisher, Phase transitions in one-dimensional cluster iteration fluids, *Ann. Phys. (N.Y.)* **58** (1970), 176–281.
10. W. Feller, Fluctuation theory of recurrent events, *Trans. Amer. Math. Soc.* **67** (1949), 98–119.
11. M. Fisher, The theory of condensation and the critical point, *Physica* **3** (1967), 255–283.
12. P. Gaspard and X.-J. Wang, Sporadicity between periodic and chaotic dynamical behaviors, *Proc. Nat. Acad. Sci. U.S.A. Physics* **85** (1988), 4591–4595.
13. F. Hofbauer, Examples for the non-uniqueness of the equilibrium states, *Trans. Amer. Math. Soc.* **228** (1977), 133–241.
14. C. Kraaikamp and A. Lopes, The theta group and the continued fraction expansion with even partial quotients, preprint, UFRGS, 1992.

15. S. P. LALLEY, Distribution of periodic orbits of symbolic and axiom A flow, *Adv. in Appl. Math.* **8** (1987), 154–193.
16. A. LOPES, The dimension spectrum of the maximal measure, *SIAM J. Math. Anal.* **20**, No. 5 (1989), 1243–1254.
17. A. LOPES, Dimension spectra and a mathematical model for phase transition, *Adv. in Appl. Math.* **11**, No. 4 (1990), 475-502.
18. A. LOPES, Entropy and large deviation, *Nonlinearity* **3**, No. 2 (1990), 527–546.
19. A. LOPES, A first order level 2 phase transition in thermodynamic formalism, *J. Statist. Phys.* **60**, Nos. 3/4 (1990), 395–411.
20. A. LOPES, Dynamics of real polynomials on the plane and triple point phase transition, *Comput. and Graphics* **16**, No. 1 (1992), 15–23.
21. R. MAÑÉ, The Hausforff dimension of invariant probabilities of rational maps, *in* "Lecture Notes in Mathematics, Vol. 1331," pp. 84–117, Springer-Verlag, New York/Berlin.
22. R. MAÑÉ, "Ergodic Theory and Differentiable Dynamics," Springer-Verlag, New York/Berlin, 1987.
23. W. PARRY, Bowen's equidistribution theory and the Dirichlet density theorem, *Ergodic Theory Dynamical Ssystems* **4** (1984), 117–134.
24. W. PARRY, An analogue of the prime number theorem for shifts of finite type and their subpensions, *Israel J. Math.* **45** (1983), 41–52.
25. W. PARRY AND M. POLLICOT, An analogue of the prime number theorem for closed orbits of axiom A flows, *Ann. of Math.* **118** (1983), 573–591.
26. W. PARRY AND M. POLLICOT, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque* (1990), 187–188.
27. M. POLLICOT, A complex Ruelle–Perron–Frobenius operator and two counter examples, *Ergodic Theory Dynamical Systems* **4** (1984), 135–147.
28. A. G. POSTNIKOV, Tauberian theorems and its applications, *Proc. Steklov. Inst. Math* (1980), Issue 2.
29. F. PRZYTYCKI, On the Perron–Frobenius–Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions, *Bol. Soc. Brasil Mat.*, to appear.
30. F. PRZYTYCKI, M. URBANSKI, AND A. ZDUNIK, Harmonic, Gibbs and Hausdorf measures on repellers for holomorphic maps, *Ann. of Math.* **130** (1989), 1–40.
31. D. RUELLE, "Thermodynamic Formalism," Addison–Wesley, Reading, MA, 1978.
32. D. RUELLE, Repellers for real analytic maps, *Ergodic Theory Dynamical Systems* **2** (1982), 99–107.
33. P. SZEPFALUSY, T. TEL, A. CSORDAS, AND Z. KOVACS, Phase transition with dynamical properties of chaotic systems, *Phys. Rev. A* **36**, No. 7 (1987), 3524–3528.
34. M. THALER, Estimates of the invariant densities of endomorphism with indifferent fixed points, *Israel J. Math.* **37**, No. 4 (1980), 303-313.
35. M. URBANSKI, Hausdorff dimension of invariant subsets for endomorphisms of the circle with an indifferent fixed point, *J. London Math. Soc.* **40**, Pt. 1 (1989), 158–170.
36. P. WALTERS, "An Intorudtion to Ergodic Theory," Springer-Verlag, New York/Berlin, 1982.
37. X.-J. WANG, Abnormal fluctuations and thermodynamic phase transition in dynamical systems, *Phys. Rev. A* **39**, No. 6 (1989).
38. X.-J. WANG, Statistical physics of temporal intermettency, *Phys. Rev. A* **40**, No. 11 (1989), 6647–6661.
39. B. WEST AND M. SHLESINGER, On the ubiquity of $1/f$ noise, *Internat. J. Modern Phys. B* **3**, No. 6 (1989), 795–819.
40. N. WIENER, The Fourier integral and certain of its applications, Dover, New York, 1933.
41. A. ZDUNIK, Parabolic orbifolds and the dimension of the maximal measure for rational maps, *Invent. Math.* **99** (1990), 627–649.