LIPSCHITZ SHADOWING FOR CONTRACTING/EXPANDING DYNAMICS ON AVERAGE

LUCAS BACKES AND DAVOR DRAGIČEVIĆ

Abstract. We prove that Lipschitz perturbations of nonautonomous contracting or expanding linear dynamics are Lipschitz shadowable provided that the Lipschitz constants are small on average. This is in sharp contrast with previous results where the Lipschitz constants are assumed to be uniformly small. Moreover, we show by means of an example that a natural extension of these results to the context of linear dynamics admitting an exponential dichotomy does not hold in general.

1. Introduction

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space, $(A_n)_{n\in\mathbb{N}}$ a sequence of bounded linear operators on X and $f_n \colon X \to X$, $n \in \mathbb{N}$ a sequence of arbitrary maps. We consider the associated (possibly) nonlinear and nonautonomous difference equation given by

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{N}. \tag{1.1}$$

We recall that (1.1) is said to exhibit *Lipschitz shadowing* (or *Hyers-Ulam stability*) if there exists $L > 0$ with the property that for each $\varepsilon > 0$ and any sequence $(y_n)_{n\in\mathbb{N}} \subset X$ with

$$\sup_{n\in\mathbb{N}} \|y_{n+1} - A_n y_n - f_n(y_n)\| \le \varepsilon,$$

there exists a solution $(x_n)_{n\in\mathbb{N}} \subset X$ of (1.1) satisfying

$$\sup_{n\in\mathbb{N}} \|x_n - y_n\| \le L\varepsilon.$$

It is proved in [1] that (1.1) is Lipschitz shadowable provided that the following holds:

(a) the sequence $(A_n)_{n\in\mathbb{N}}$ admits an exponential dichotomy (see Definition 4.1);
(b) there exists a sufficiently small $c > 0$ such that

$$\|f_n(x) - f_n(y)\| \le c\|x - y\|, \quad x, y \in X, \ n \in \mathbb{N}. \tag{1.2}$$

For related results which deal with the case when $f_n \equiv 0$ and the sequence $(A_n)_{n\in\mathbb{N}}$ is either constant or periodic, we refer to [5, 7, 8, 9] and references therein. Moreover, in [2, 14] one can find related results without any periodicity assumptions on $(A_n)_{n\in\mathbb{N}}$ and without the requirement that maps f_n vanish. Finally, Lipschitz shadowing of nonlinear and nonautonomous difference equations with *delay* is discussed in [4, 10]. For a detailed survey on shadowing in the context of smooth dynamics we refer to [12, 13].

It is natural to ask whether it is possible to ensure that (1.1) exhibits Lipschitz shadowing by relaxing condition (b), i.e. without requiring that nonlinearities f_n are uniformly Lipschitz with a sufficiently small Lipschitz constant.

2020 *Mathematics Subject Classification.* Primary: 37C50; Secondary: 34D10, 39A30.
Key words and phrases. Lipschitz shadowing, nonautonomous linear dynamics, Lipschitz perturbations, smallnes on average.

The main objective of the present paper is to formulate positive results in this direction. More precisely, we prove (see Theorems 2.4 and 3.2) that (1.1) exhibits Lipschitz shadowing provided that the sequence $(A_n)_{n\in\mathbb{N}}$ is either exponentially stable or expanding, and that each f_n is a Lipschitz map with a Lipschitz constant p_n, where the sequence $(p_n)_{n\in\mathbb{N}} \subset (0,\infty)$ satisfies only certain smallness in average condition (see (2.4) and (3.3)). These conditions allow that for some values of n, p_n can attain arbitrary large values.

In fact, our results are more general and deal with the case when the sequence $(A_n)_{n\in\mathbb{N}}$ is *nonuniformly* exponentially stable or expanding, yielding shadowing results in the spirit of those discussed in [2]. In addition, by constructing an explicit example (see Example 4.3) we illustrate that our assumptions on the sequence $(A_n)_{n\in\mathbb{N}}$ cannot be relaxed. More precisely, in the case when $(A_n)_{n\in\mathbb{N}}$ admits an exponential dichotomy (with both stable and unstable subspaces being nontrivial), it is impossible to deduce Lipschitz shadowability of (1.1) under only "smallness in average" condition for the sequence $(p_n)_{n\in\mathbb{N}}$.

Consequently, in comparison with our previously described results from [1] we are able to substantially relax the condition (1.2). On the other hand, we impose a more restrictive condition on the sequence $(A_n)_{n\in\mathbb{N}}$ by requiring that it admits an exponential dichotomy with either stable or unstable subspaces being trivial.

Finally, we stress that besides considering the case of discrete time, we also deal with the case of continuous time (see Theorems 2.7 and 3.7).

Besides our previous work, our arguments are inspired by the work of Nam [11] who obtained similar results in the one-dimensional case (i.e. $X = \mathbb{R}$ or $X = \mathbb{C}$).

2. The contracting on average case

In this section we consider the case of dynamics that is contracting on average. All throughout the paper, let $X = (X, \|\cdot\|)$ be an arbitrary Banach space and denote by $\mathcal{B}(X)$ the space of all bounded linear operators equipped with the operator norm which we also denote by $\|\cdot\|$.

2.1. The discrete time case.

Given a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(X)$, let us consider the associated linear difference equation

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}. \tag{2.1}$$

For $m, n \in \mathbb{N}$, the evolution operator associated to (2.1) is given by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & m > n; \\ \mathrm{Id} & m = n, \end{cases}$$

where Id denotes the identity operator on X.

Definition 2.1. Let $\boldsymbol{\nu} = (\nu_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. We say that (2.1) or that the sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ is *$\boldsymbol{\nu}$-nonuniformly exponentially stable* if there exist $D, \lambda > 0$ such that

$$\|\mathcal{A}(m,n)\| \le D\nu_n e^{-\lambda(m-n)}, \quad m \ge n. \tag{2.2}$$

Remark 2.2. We note that the classical concept of *(uniform) exponential stability* corresponds to the choice $\boldsymbol{\nu} = (\nu_n)_{n\in\mathbb{N}}$, where $\nu_n = 1$ for each $n \in \mathbb{N}$.

Example 2.3. Let $X = \mathbb{R}$, choose $\omega < 0$ and $\varepsilon \ge 0$, and set

$$A_n := e^{\omega+\varepsilon[(-1)^n n - 1/2]}, \quad n \in \mathbb{N}.$$

Then, (2.1) is $\boldsymbol{\nu}$-nonuniformly exponentially stable with $\nu_n = e^{\varepsilon n}$, $n \in \mathbb{N}$. Moreover, if $\varepsilon > 0$ then (2.1) is not (uniformly) exponentially stable. We refer to [6, Example 1] for details.

Theorem 2.4. *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ be a sequence that is $\boldsymbol{\nu}$-nonuniformly stable. Moreover, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps $f_n\colon X \to X$ with the property that there is a sequence $(p_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that*

$$\|f_n(x) - f_n(y)\| \le \frac{p_n}{\nu_{n+1}}\|x-y\| \quad \text{for } n \in \mathbb{N} \text{ and } x, y \in X. \tag{2.3}$$

Finally, we assume that

$$\lim_{n\to\infty}\left(\prod_{i=0}^{n-1}\left(e^{-\lambda}+Dp_i\right)\right)^{\frac{1}{n}} = \rho \quad \text{for some } \rho \in (0,1), \tag{2.4}$$

where $D, \lambda > 0$ are such that (2.2) holds. Then, there exists a constant $L > 0$ such that for each $\varepsilon > 0$ and any sequence $(y_n)_{n\in\mathbb{N}} \subset X$ satisfying

$$\|y_{n+1} - A_n y_n - f_n(y_n)\| \le \frac{\varepsilon}{\nu_{n+1}} \quad n \in \mathbb{N}, \tag{2.5}$$

there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with the properties that

$$x_{n+1} = A_n x_n + f_n(x_n) \quad n \in \mathbb{N}, \tag{2.6}$$

and

$$\sup_{n\in\mathbb{N}}\|x_n - y_n\| \le L\varepsilon. \tag{2.7}$$

In particular, when $\sup_{n\in\mathbb{N}}\nu_n < \infty$ we have that (2.6) is Lipschitz shadowable.

Proof. For $n \in \mathbb{N}$ and $x \in X$, let

$$\|x\|_n := \sup_{m\ge n}\left(\|\mathcal{A}(m,n)x\|e^{\lambda(m-n)}\right).$$

By (2.2) we have that

$$\|x\| \le \|x\|_n \le D\nu_n\|x\|, \quad \text{for } n \in \mathbb{N} \text{ and } x \in X. \tag{2.8}$$

Observe that

$$\begin{aligned}
\|\mathcal{A}(m,n)x\|_m &= \sup_{k\ge m}\left(\|\mathcal{A}(k,m)\mathcal{A}(m,n)x\|e^{\lambda(k-m)}\right)\\
&= e^{-\lambda(m-n)}\sup_{k\ge m}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(k-n)}\right)\\
&\le e^{-\lambda(m-n)}\sup_{k\ge n}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(k-n)}\right)\\
&\le e^{-\lambda(m-n)}\|x\|_n,
\end{aligned}$$

for $m \ge n$ and $x \in X$. We conclude that

$$\|\mathcal{A}(m,n)x\|_m \le e^{-\lambda(m-n)}\|x\|_n, \quad \text{for } m \ge n \text{ and } x \in X.$$

In particular,

$$\|A_n x\|_{n+1} \le e^{-\lambda}\|x\|_n, \quad \text{for } n \in \mathbb{N} \text{ and } x \in X. \tag{2.9}$$

Let $(y_n)_{n\in\mathbb{N}} \subset X$ be a sequence satisfying (2.5) for some $\varepsilon > 0$. We define a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ recursively. Namely, we set $x_0 := y_0$ and $x_{n+1} = A_n x_n + f_n(x_n)$ for $n \in \mathbb{N}$.

We claim that

$$\|x_n - y_n\|_n \le D\varepsilon\sum_{j=0}^{n-1}\prod_{i=j+1}^{n-1}(e^{-\lambda}+Dp_i), \quad n \in \mathbb{N}, \tag{2.10}$$

with the convention that $\prod_i^j := 1$ if $j < i$. For $n = 0$ there is nothing to prove. Suppose that (2.10) holds for $n \in \mathbb{N}$. By (2.3), (2.5), (2.8) and (2.9) we have that

$$\begin{aligned}
&\|x_{n+1} - y_{n+1}\|_{n+1} \\
&= \|A_n x_n + f_n(x_n) - y_{n+1}\|_{n+1} \\
&\le \|A_n(x_n - y_n) + f_n(x_n) - f_n(y_n)\|_{n+1} + \|A_n y_n + f_n(y_n) - y_{n+1}\|_{n+1} \\
&\le e^{-\lambda}\|x_n - y_n\|_n + \|f_n(x_n) - f_n(y_n)\|_{n+1} + \|A_n y_n + f_n(y_n) - y_{n+1}\|_{n+1} \\
&\le e^{-\lambda}\|x_n - y_n\|_n + D\nu_{n+1}\|f_n(x_n) - f_n(y_n)\| + D\nu_{n+1}\|A_n y_n + f_n(y_n) - y_{n+1}\| \\
&\le e^{-\lambda}\|x_n - y_n\|_n + Dp_n\|x_n - y_n\| + D\varepsilon \\
&\le (e^{-\lambda} + Dp_n)\|x_n - y_n\|_n + D\varepsilon \\
&\le D\varepsilon(e^{-\lambda} + Dp_n)\sum_{j=0}^{n-1}\prod_{i=j+1}^{n-1}(e^{-\lambda} + Dp_i) + D\varepsilon \\
&= D\varepsilon\sum_{j=0}^{n-1}\prod_{i=j+1}^{n}(e^{-\lambda} + Dp_i) + D\varepsilon \\
&= D\varepsilon\sum_{j=0}^{n}\prod_{i=j+1}^{n}(e^{-\lambda} + Dp_i),
\end{aligned}$$

yielding (2.10). On the other hand, in the proof of [11, Theorem 2.4] it is proved that (2.4) implies that there exists $c > 0$ such that

$$\sum_{j=0}^{n}\prod_{i=j+1}^{n}(e^{-\lambda} + Dp_i) \le c, \quad n \in \mathbb{N}.$$

Hence, (2.8) combined with (2.10) gives that

$$\|x_n - y_n\| \le \|x_n - y_n\|_n \le cD\varepsilon \quad n \in \mathbb{N}.$$

Therefore, setting $L := cD > 0$ we conclude that (2.7) holds. □

Remark 2.5. We note that (2.4) is trivially satisfied if $p_n = c$ with $c \ge 0$ and $e^{-\lambda} + Dc < 1$. In this case, Theorem 2.4 follows from [1, Theorem 5] when $\nu_n = 1$, $n \in \mathbb{N}$.

Let us now illustrate that Theorem 2.4 is more general. For this purpose, fix arbitrary $\rho \in (e^{-\lambda}, 1)$ and $C > 0$. We define a sequence $(p_n)_{n\in\mathbb{N}}$ by

$$p_n = \begin{cases} \frac{\rho - e^{-\lambda}}{D} & \text{if } n \text{ is not of the form } 10^k \text{ for } k \in \mathbb{N}; \\ C & \text{if } n = 10^k \text{ for some } k \in \mathbb{N}. \end{cases}$$

It is straightforward to verify that (2.4) holds. On the other hand, clearly, if C is such that $e^{-\lambda} + DC \ge 1$, we do not have that $\sup_{n\in\mathbb{N}}(e^{-\lambda} + Dp_n) < 1$.

2.2. The continuous time case.

Let us now discuss the case of continuous time. Let $A\colon [0, \infty) \to \mathcal{B}(X)$ be a continuous map and consider the associated linear nonautonomous differential equation given by

$$x' = A(t)x, \quad t \ge 0. \tag{2.11}$$

By $T(t, s)$ we will denote the evolution family associated to (2.11).

Definition 2.6. Let $\nu\colon [0, \infty) \to (0, \infty)$ be an arbitrary function. We say that (2.11) is *ν-nonuniformly exponentially stable* if there exist $D, \lambda > 0$ such that

$$\|T(t, s)\| \le D\nu(s)e^{-\lambda(t-s)}, \quad t \ge s \ge 0. \tag{2.12}$$

Theorem 2.7. *Suppose that* (2.11) *is* ν*-nonuniformly exponentially stable. Moreover, let* $f\colon [0,\infty)\times X\to X$ *be a continuous map with the property that there is a locally integrable function* $p\colon [0,\infty)\to(0,\infty)$ *such that*

$$\|f(t,x)-f(t,y)\|\le \frac{p(t)}{\nu(t)}\|x-y\|, \quad x,y\in X,\ t\ge 0. \tag{2.13}$$

Set $p_n:=\int_n^{n+1}p(t)\,dt$ *for* $n\in\mathbb{N}$ *and suppose that* $\mathcal{P}:=\sup_{n\in\mathbb{N}}e^{p_n}<+\infty$ *and*

$$\lim_{n\to\infty}\left(\prod_{i=0}^{n-1}\left(e^{-\lambda}+D\mathcal{P}^D p_i\right)\right)^{\frac{1}{n}}=\rho \quad \text{for some } \rho\in(0,1), \tag{2.14}$$

where $D,\lambda>0$ *are such that* (2.12) *holds. Then, there is a constant* $K>0$ *such that for each* $\varepsilon>0$ *and any continuously differentiable function* $y\colon[0,\infty)\to X$ *satisfying*

$$\|y'(t)-A(t)y(t)-f(t,y(t))\|\le\frac{\varepsilon}{\nu(t)} \quad t\ge 0, \tag{2.15}$$

there exists a function $x\colon[0,\infty)\to X$ *such that*

$$x'(t)=A(t)x(t)+f(t,x(t)) \quad t\ge 0, \tag{2.16}$$

and

$$\sup_{t\ge 0}\|x(t)-y(t)\|\le K\varepsilon. \tag{2.17}$$

In particular, if $\sup_{t\ge0}\nu(t)<+\infty$*, then the differential equation* (2.16) *is Lipschitz-shadowable.*

Proof. By $U(t,s)$ we will denote the nonlinear evolution family corresponding to (2.16), i.e. $t\mapsto U(t,s)v$ is the unique solution of (2.16) with value v at time s. By the variation of constants formula we have that

$$U(t,s)x=T(t,s)x+\int_s^t T(t,r)f(r,U(r,s)x)\,dr, \quad \text{for } t\ge s \text{ and } x\in X. \tag{2.18}$$

Set

$$A_n:=T(n+1,n), \quad n\in\mathbb{N}.$$

For $s\ge 0$ and $x\in X$, let

$$\|x\|_s:=\sup_{t\ge s}\left(\|T(t,s)x\|e^{\lambda(t-s)}\right).$$

Then,

$$\|x\|\le\|x\|_s\le D\nu(s)\|x\|, \quad \text{for } x\in X \text{ and } s\ge 0. \tag{2.19}$$

Moreover, one can easily show that

$$\|T(t,s)x\|_t\le e^{-\lambda(t-s)}\|x\|_s, \quad t\ge s\ge 0. \tag{2.20}$$

Observe that it follows from (2.13), (2.18), (2.19) and (2.20) that for $x,y\in X$ and $t\ge s\ge 0$,

$$\begin{aligned}\|U(t,s)x-U(t,s)y\|_t &\le \|T(t,s)(x-y)\|_t\\ &\quad+\int_s^t\|T(t,r)(f(r,U(r,s)x)-f(r,U(r,s)y))\|_t\,dr\\ &\le e^{-\lambda(t-s)}\|x-y\|_s\\ &\quad+\int_s^t e^{-\lambda(t-r)}\|f(r,U(r,s)x)-f(r,U(r,s)y)\|_r\,dr\\ &\le e^{-\lambda(t-s)}\|x-y\|_s\\ &\quad+D\int_s^t e^{-\lambda(t-r)}p(r)\|U(r,s)x-U(r,s)y\|_r\,dr.\end{aligned}$$

By Gronwall's lemma, we conclude that

$$\|U(t,s)x-U(t,s)y\|_t \le e^{D\int_s^t p(r)\,dr}\|x-y\|_s, \quad \text{for } t\ge s\ge 0 \text{ and } x,y\in X. \tag{2.21}$$

For $n\in\mathbb{N}$ and $x\in X$, set

$$f_n(x) := \int_n^{n+1} T(n+1,r)f(r,U(r,n)x)\,dr. \tag{2.22}$$

Then, (2.13), (2.19), (2.20) and (2.21) give that

$$\begin{aligned}\|f_n(x)-f_n(y)\|_{n+1} &\le \int_n^{n+1} \|T(n+1,r)(f(r,U(r,n)x)-f(r,U(r,n)y))\|_{n+1}\,dr \\ &\le \int_n^{n+1} e^{-\lambda(n+1-r)}\|f(r,U(r,n)x)-f(r,U(r,n)y)\|_r\,dr \\ &\le D\int_n^{n+1} p(r)\|U(r,n)x-U(r,n)y\|_r\,dr \\ &\le D\|x-y\|_n e^{D\int_n^{n+1}p(r)\,dr}\int_n^{n+1} p(r)\,dr,\end{aligned}$$

which implies that

$$\|f_n(x)-f_n(y)\|_{n+1} \le D\mathcal{P}^D p_n\|x-y\|_n, \quad \text{for } n\in\mathbb{N} \text{ and } x,y\in X.$$

Furthermore, by (2.18) we have that $A_n+f_n=U(n+1,n)$ for every $n\in\mathbb{N}$.

Let $\varepsilon>0$ be arbitrary and choose a continuously differentiable function $y\colon[0,\infty)\to X$ satisfying (2.15). Fix $n\in\mathbb{N}$ arbitrary and let $\bar{y}\colon[n,\infty)\to X$ be the solution of (2.16) on $[n,\infty)$ such that $\bar{y}(n)=y(n)$. Then,

$$y'(t)-\bar{y}'(t)=A(t)(y(t)-\bar{y}(t))+f(t,y(t))-f(t,\bar{y}(t))+z(t)$$

with $z(t):=y'(t)-A(t)y(t)-f(t,y(t))$ and, consequently,

$$\begin{aligned}y(t)-\bar{y}(t) &= T(t,n)(y(n)-\bar{y}(n))+\int_n^t T(t,r)(f(r,y(r))-f(r,\bar{y}(r)))\,dr \\ &\quad+\int_n^t T(t,r)z(r) \\ &= \int_n^t T(t,r)(f(r,y(r))-f(r,\bar{y}(r)))\,dr+\int_n^t T(t,r)z(r)\end{aligned} \tag{2.23}$$

for $t\ge n$. Hence, by (2.13), (2.15), (2.19) and (2.20) we have that

$$\begin{aligned}\|y(t)-\bar{y}(t)\|_t &\le D\int_n^t e^{-\lambda(t-r)}p(r)\|y(r)-\bar{y}(r)\|_r\,dr+\int_n^t e^{-\lambda(t-r)}\|z(r)\|_r\,dr \\ &\le D\varepsilon+D\int_n^t p(r)\|y(r)-\bar{y}(r)\|_r\,dr,\end{aligned}$$

for $n\in\mathbb{N}$ and $t\in[n,n+1]$. By applying Gronwall's lemma we conclude that

$$\|y(t)-\bar{y}(t)\|_t \le D\mathcal{P}^D\varepsilon \quad \text{for } n\in\mathbb{N},\ t\in[n,n+1],$$

which in particular implies that (recall that $A_n+f_n=U(n+1,n)$)

$$\|y(n+1)-A_ny(n)-f_n(y(n))\|_{n+1}\le D\mathcal{P}^D\varepsilon, \quad n\in\mathbb{N}.$$

It follows from the proof of the previous theorem that the sequence $(x_n)_{n\in\mathbb{N}}\subset X$ given by $x_0=y(0)$ and $x_{n+1}=A_nx_n+f_n(x_n)$ for $n\in\mathbb{N}$ satisfies

$$\|x_n-y(n)\|_n\le L\varepsilon \quad n\in\mathbb{N}, \tag{2.24}$$

where $L>0$ depends only on D,λ and the sequence $(p_n)_n$. We set

$$x(t):=U(t,n)x_n, \quad \text{for } n\in\mathbb{N} \text{ and } t\in[n,n+1). \tag{2.25}$$

Then, x is a solution of (2.16). For $n \in \mathbb{N}$ and $t \in [n, n+1)$ we have that

$$x'(t) - y'(t) = A(t)(x(t) - y(t)) + f(t, x(t)) - f(t, y(t)) - z(t)$$

where $z(t)$ is as defined above. Consequently,

$$\begin{aligned} x(t) - y(t) = T(t,n)(x(n) - y(n)) + \int_n^t T(t,r)(f(r,x(r)) - f(r,y(r)))\,dr \\ - \int_n^t T(t,r)z(r)\,dr, \end{aligned} \tag{2.26}$$

for $t \in [n, n+1]$. From this we get (using (2.13), (2.15), (2.19), (2.20) and (2.24)) that

$$\begin{aligned} \|x(t) - y(t)\|_t &\le \|x_n - y(n)\|_n + D\varepsilon + D\int_n^t p(r)\|x(r) - y(r)\|_r\,dr \\ &\le (D+L)\varepsilon + D\int_n^t p(r)\|x(r) - y(r)\|_r\,dr, \end{aligned}$$

for $t \in [n, n+1]$. Finally, Gronwall's lemma gives us that

$$\|x(t) - y(t)\| \le \|x(t) - y(t)\|_t \le (D+L)\mathcal{P}^D\varepsilon,$$

yielding that (2.17) holds with $K := (D+L)\mathcal{P}^D > 0$. The proof of the theorem is completed. ☐

3. The expanding on average case

In this section we consider the case of dynamics that is expanding on average.

3.1. **The discrete time case.** Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of *invertible* bounded linear operators in $\mathcal{B}(X)$ and consider the associated linear difference equation (2.1). In this case, besides considering the evolution operator $\mathcal{A}(m,n)$ for $m \ge n$, we may also consider

$$\mathcal{A}(m,n) = A_m^{-1}A_{m+1}^{-1}\cdots A_{n-1}^{-1}$$

for $m < n$.

Definition 3.1. Let $\boldsymbol{\nu} = (\nu_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. We say that (2.1) or that the sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ is *$\boldsymbol{\nu}$-nonuniformly exponentially expanding* if there exist $D, \lambda > 0$ such that

$$\|\mathcal{A}(m,n)\| \le D\nu_n e^{-\lambda(n-m)} \quad \text{for } m \le n. \tag{3.1}$$

Theorem 3.2. *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ be a sequence which is $\boldsymbol{\nu}$-nonuniformly exponentially expanding. Moreover, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps $f_n\colon X \to X$ such that $A_n + f_n$ is surjective for every $n \in \mathbb{N}$ and with the property that there is a sequence $(p_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that*

$$\|f_n(x) - f_n(y)\| \le \frac{p_n}{\nu_{n+1}}\|x - y\| \quad \text{for } n \in \mathbb{N} \text{ and } x, y \in X. \tag{3.2}$$

Finally, we assume that

$$\lim_{n\to\infty}\left(\prod_{i=0}^{n-1}\left(e^\lambda - Dp_i\right)\right)^{\frac{1}{n}} = \rho \quad \text{for some } \rho \in (1,\infty), \tag{3.3}$$

where $D, \lambda > 0$ are such that (3.1) holds. Then, there exists a constant $L > 0$ such that for each $\varepsilon > 0$ and any sequence $(y_n)_{n\in\mathbb{N}} \subset X$ satisfying

$$\|y_{n+1} - A_n y_n - f_n(y_n)\| \le \frac{\varepsilon}{\nu_{n+1}} \quad n \in \mathbb{N}, \tag{3.4}$$

there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ *with the properties that*

$$x_{n+1} = A_n x_n + f_n(x_n) \quad n \in \mathbb{N}, \tag{3.5}$$

and

$$\sup_{n\in\mathbb{N}} \|x_n - y_n\| \le L\varepsilon. \tag{3.6}$$

In particular, if $\sup_{n\in\mathbb{N}} \nu_n < +\infty$, *then the difference equation* (3.5) *is Lipschitz shadowable.*

Proof. For $x \in X$ and $n \in \mathbb{N}$, set

$$\|x\|_n := \sup_{m\le n} \left(\|\mathcal{A}(m,n)x\| e^{\lambda(n-m)}\right).$$

Then, using (3.1), we get that for every $x \in X$ and $n \in \mathbb{N}$,

$$\|x\| \le \|x\|_n \le D\nu_n \|x\|. \tag{3.7}$$

Moreover,

$$\|\mathcal{A}(m,n)x\|_m \le e^{-\lambda(n-m)}\|x\|_n,$$

for $m \le n$ and $x \in X$. In particular,

$$\|x\|_n \le e^{-\lambda}\|A_n x\|_{n+1}$$

for every $n \in \mathbb{N}$ and $x \in X$. Furthermore, we observe that

$$\|f_n(x) - f_n(y)\|_{n+1} \le D\nu_{n+1}\|f_n(x) - f_n(y)\| \le Dp_n\|x - y\| \le Dp_n\|x - y\|_n,$$

for $x, y \in X$ and $n \in \mathbb{N}$. Hence, making $F_n := A_n + f_n$ and $q_n := e^{\lambda} - Dp_n$ we have that

$$\begin{aligned}\|F_n(x) - F_n(y)\|_{n+1} &\ge \|A_n x - A_n y\|_{n+1} - \|f_n(x) - f_n(y)\|_{n+1} \\ &\ge \left(e^{\lambda} - Dp_n\right)\|x - y\|_n \\ &= q_n\|x - y\|_n,\end{aligned}$$

for $x, y \in X$ and $n \in \mathbb{N}$. That is,

$$\|F_n(x) - F_n(y)\|_{n+1} \ge q_n\|x - y\|_n, \tag{3.8}$$

for $x, y \in X$ and $n \in \mathbb{N}$.

Given $\varepsilon > 0$, let $(y_n)_{n\in\mathbb{N}} \subset X$ be a sequence satisfying (3.4). For each $n \in \mathbb{N}$, let us take $z_n \in X$ such that

$$F_{n-1} \circ \ldots \circ F_0(z_n) = y_n.$$

We claim that the sequence $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Indeed, given $n, k \in \mathbb{N}$, it follows by (3.8) that

$$\|z_n - z_{n+k}\|_0 \le \left(\prod_{j=0}^{n-1} \frac{1}{q_j}\right) \|y_n - F_{n-1} \circ \ldots \circ F_0(z_{n+k})\|_n.$$

Now, combining (3.7), (3.8) and (3.4), we obtain that

$$\begin{aligned}&\|y_n - F_{n-1} \circ \ldots \circ F_0(z_{n+k})\|_n \\ &\le \|y_n - w_n\|_n + \|w_n - F_{n-1} \circ \ldots \circ F_n(z_{n+k})\|_n \\ &\le \frac{D\varepsilon}{q_n} + \frac{1}{q_n}\|y_{n+1} - F_n \circ \ldots \circ F_0(z_{n+k})\|_{n+1},\end{aligned}$$

where $w_n \in X$ is such that $F_n(w_n) = y_{n+1}$. Using this fact and proceeding recursively we get that

$$\|y_n - F_{n-1} \circ \ldots \circ F_0(z_{n+k})\|_n \le D\varepsilon \sum_{j=n}^{n+k-1} \left(\prod_{i=n}^{j} \frac{1}{q_i}\right). \tag{3.9}$$

Consequently, combining these observations we conclude that

$$\begin{aligned}\|z_n - z_{n+k}\|_0 &\le D\varepsilon \left(\prod_{j=0}^{n-1} \frac{1}{q_j}\right) \sum_{j=n}^{n+k-1} \left(\prod_{i=n}^{j} \frac{1}{q_i}\right) \\ &= D\varepsilon \sum_{j=n}^{n+k-1} \left(\prod_{i=0}^{j} \frac{1}{q_i}\right) \\ &\le D\varepsilon \sum_{j=n}^{+\infty} \left(\prod_{i=0}^{j} \frac{1}{q_i}\right).\end{aligned} \tag{3.10}$$

Finally, observe that condition (3.3) is equivalent to

$$\lim_{n\to+\infty} \left(\prod_{i=0}^{n} \frac{1}{q_i}\right)^{\frac{1}{n}} < 1.$$

Thus, by the root test we have that the series $\sum_{j=0}^{+\infty} \left(\prod_{i=0}^{j} \frac{1}{q_i}\right)$ converges and, in particular,

$$\sum_{j=n}^{+\infty} \left(\prod_{i=0}^{j} \frac{1}{q_i}\right) \xrightarrow{n\to+\infty} 0.$$

Then, it follows by (3.10) that $(z_n)_{n\in\mathbb{N}}$ is indeed a Cauchy sequence as claimed and, in particular, since X is a Banach space, there exists $x_0 \in X$ such that $x_0 = \lim\limits_{n\to+\infty} z_n$.

Let us consider now the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} = F_n(x_n)$ for $n \in \mathbb{N}$. We claim that there exists $C > 0$ (independent on $(y_n)_n$) such that

$$\sup_{n\in\mathbb{N}} \|x_n - y_n\| \le C\varepsilon.$$

In fact, given $n \in \mathbb{N}$, by (3.7) and (3.9) it follows that for any $k \in \mathbb{N}$,

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - F_{n-1}\circ\ldots\circ F_0(z_{n+k})\| + \|F_{n-1}\circ\ldots\circ F_0(z_{n+k}) - y_n\| \\ &\le \|x_n - F_{n-1}\circ\ldots\circ F_0(z_{n+k})\| + \|F_{n-1}\circ\ldots\circ F_0(z_{n+k}) - y_n\|_n \\ &\le \|x_n - F_{n-1}\circ\ldots\circ F_0(z_{n+k})\| + D\varepsilon \sum_{j=n}^{n+k-1} \left(\prod_{i=n}^{j} \frac{1}{q_i}\right).\end{aligned}$$

Now we need the following auxiliary result.

Lemma 3.3. *There exists $\tilde{C} > 0$ such that $\sum_{j=n}^{n+k-1} \left(\prod_{i=n}^{j} \frac{1}{q_i}\right) \le \tilde{C}$ for every $n, k \in \mathbb{N}$.*

Proof. The proof of this result can be obtained by proceeding as in the proof of [11, Theorem 3.4]. In fact, from Eq. (3.3) on, the proof of the aforementioned result consists precisely in showing that $\sum_{j=n}^{+\infty} \left(\prod_{i=n}^{j} \frac{1}{q_i}\right) \le \tilde{C}$ for some $\tilde{C} > 0$ and every $n \in \mathbb{N}$. ☐

Thus, taking $\tilde{C} > 0$ as in Lemma 3.3 and $C = \tilde{C}D$ it follows that

$$\|x_n - y_n\| \le \|x_n - F_{n-1}\circ\ldots\circ F_0(z_{n+k})\| + C\varepsilon \tag{3.11}$$

for every $k \in \mathbb{N}$. Moreover,

$$\begin{aligned}&\|x_n - F_{n-1}\circ\ldots\circ F_0(z_{n+k})\| \\ &= \|F_{n-1}\circ\ldots\circ F_0(x_0) - F_{n-1}\circ\ldots\circ F_0(z_{n+k})\|.\end{aligned}$$

Therefore, since each F_j is continuous and $\lim_{k\to+\infty} z_{n+k} = x_0$, it follows that

$$\|x_n - F_{n-1}\circ \ldots \circ F_0(z_{n+k})\| \xrightarrow{k\to+\infty} 0.$$

Consequently, taking $k \to +\infty$ in (3.11) we conclude that

$$\sup_{n\in\mathbb{N}} \|x_n - y_n\| \le C\varepsilon$$

as claimed. The proof of the theorem is complete. □

Remark 3.4. We observe that whenever A_n is invertible and $p_n \le \|A_n^{-1}\|^{-1}$ we have that $A_n + f_n$ is a homeomorphism and, in particular, is surjective (see [3, Remark 3.1]). On the other hand, this is obviously not a necessary condition to guarantee that $A_n + f_n$ is surjective. For instance, again if A_n is invertible, then taking $f_n = cA_n$ for any $c \ne -1$ we see that $A_n + f_n$ is surjective while, in this case, $p_n = |c|\|A_n\|$ can be arbitrary large.

Remark 3.5. As in Remark 2.5, we observe that (3.3) is trivially satisfied if $p_n = c$ with $c \ge 0$ and $e^\lambda - Dc > 1$. In this case, Theorem 3.2 follows from [1, Theorem 5] when $\nu_n = 1$, $n \in \mathbb{N}$. Moreover, similarly to what we did in the above-mentioned remark we can also construct an example that satisfies (3.3) which does not satisfy $\inf_{n\in\mathbb{N}}(e^\lambda - Dp_n) > 1$.

3.2. **The continuous time case.** In this section we obtain a continuous time version of the results from Section 3.1. For this purpose, let us consider the differential equation (2.11) and, as before, denote by $T(t,s)$ the evolution family associated to it.

Definition 3.6. Let $\nu\colon [0,\infty) \to (0,\infty)$ be an arbitrary function. We say that (2.11) is *ν-nonuniformly exponentially expanding* if there exist $D, \lambda > 0$ such that

$$\|T(t,s)\| \le D\nu(s)e^{-\lambda(s-t)}, \quad 0 \le t \le s. \tag{3.12}$$

Theorem 3.7. *Suppose that (2.11) is ν-nonuniformly exponentially expanding, and that there exist $D' > 0$ and $a \ge \lambda$ such that*

$$\|T(t,s)\| \le D'\nu(s)e^{a(t-s)}, \quad t \ge s \ge 0. \tag{3.13}$$

Moreover, let $f\colon [0,\infty)\times X \to X$ be a continuous map with the property that there is a locally integrable function $p\colon [0,\infty) \to (0,\infty)$ such that

$$\|f(t,x) - f(t,y)\| \le \frac{p(t)}{\nu(t)}\|x-y\|, \quad x, y \in X,\ t \ge 0. \tag{3.14}$$

Set $p_n := \int_n^{n+1} p(t)\,dt$ for $n \in \mathbb{N}$ and suppose that $\mathcal{P} := \sup_{n\in\mathbb{N}} e^{p_n} < +\infty$ and

$$\lim_{n\to\infty}\left(\prod_{i=0}^{n-1}\left(2e^\lambda - 4D''e^{2a}\mathcal{P}^{2D''}p_i\right)\right)^{\frac{1}{n}} = \rho \quad \textit{for some } \rho \in (1,\infty), \tag{3.15}$$

where $D'' := D + D'$ and $D, \lambda > 0$ are such that (3.12) holds. Then, there is a constant $K > 0$ such that for each $\varepsilon > 0$ and any continuously differentiable function $y\colon [0,\infty) \to X$ satisfying

$$\|y'(t) - A(t)y(t) - f(t,y(t))\| \le \frac{\varepsilon}{\nu(t)} \quad t \ge 0, \tag{3.16}$$

there exists a function $x\colon [0,\infty) \to X$ such that

$$x'(t) = A(t)x(t) + f(t,x(t)) \quad t \ge 0, \tag{3.17}$$

and

$$\sup_{t\ge 0}\|x(t) - y(t)\| \le K\varepsilon. \tag{3.18}$$

In particular, if $\sup_{t\ge 0}\nu(t) < +\infty$*, then the differential equation* (2.16) *is Lipschitz shadowable.*

Proof. For $s \ge 0$ and $x \in X$, let

$$\|x\|_s := \sup_{t\le s}\left(\|T(t,s)x\|e^{\lambda(s-t)}\right) + \sup_{t>s}\left(\|T(t,s)x\|e^{-a(t-s)}\right).$$

Then,

$$\|x\| \le \|x\|_s \le D''\nu(t)\|x\|, \quad \text{for } s\ge 0 \text{ and } x\in X. \tag{3.19}$$

Moreover, for $0\le t\le s$ and $x \in X$ we have that

$$\begin{aligned}
\|T(t,s)x\|_t &= \sup_{r\le t}\left(\|T(r,s)x\|e^{\lambda(t-r)}\right) + \sup_{r>t}\left(\|T(r,s)x\|e^{-a(r-t)}\right)\\
&\le e^{-\lambda(s-t)}\sup_{r\le t}\left(\|T(r,s)x\|e^{\lambda(s-r)}\right) + e^{-a(s-t)}\sup_{t<r\le s}\left(\|T(r,s)x\|e^{a(s-r)}\right)\\
&\quad + e^{-a(t-s)}\sup_{r>s}\left(\|T(r,s)x\|e^{-a(r-s)}\right)\\
&\le 2e^{-\lambda(s-t)}\sup_{r\le s}\left(\|T(r,s)x\|e^{\lambda(s-r)}\right) + e^{-\lambda(s-t)}\sup_{r>s}\left(\|T(r,s)x\|e^{-a(r-s)}\right)\\
&\le 2e^{-\lambda(s-t)}\|x\|_s.
\end{aligned}$$

Hence,

$$\|T(t,s)x\|_t \le 2e^{-\lambda(s-t)}\|x\|_s, \quad \text{for } 0\le t\le s \text{ and } x\in X.$$

Similarly one can show that

$$\|T(t,s)x\|_t \le 2e^{a(t-s)}\|x\|_s, \quad \text{for } t\ge s\ge 0 \text{ and } x \in X. \tag{3.20}$$

Let $U(t,s)$ be given by (2.18). It follows from (3.14), (3.19) and (3.20) that for $x,y\in X$ and $t\ge s\ge 0$, we have that

$$\begin{aligned}
\|U(t,s)x-U(t,s)y\|_t &\le \|T(t,s)(x-y)\|_t\\
&\quad + \int_s^t \|T(t,r)(f(r,U(r,s)x)-f(r,U(r,s)y))\|_t\,dr\\
&\le 2e^{a(t-s)}\|x-y\|_s\\
&\quad + 2\int_s^t e^{a(t-r)}\|f(r,U(r,s)x)-f(r,U(r,s)y)\|_r\,dr\\
&\le 2e^{a(t-s)}\|x-y\|_s\\
&\quad + 2D''\int_s^t e^{a(t-r)}p(r)\|U(r,s)x-U(r,s)y\|_r\,dr.
\end{aligned}$$

By Gronwall's lemma, we conclude that

$$\|U(t,s)x-U(t,s)y\|_t \le 2e^{a(t-s)+2D''\int_s^t p(r)\,dr}\|x-y\|_s, \tag{3.21}$$

for $t\ge s\ge 0$ and $x,y\in X$. Let $f_n\colon X\to X$, $n\in\mathbb{N}$ be given by (2.22). Then, (3.14), (3.20)

$$\begin{aligned}
\|f_n(x)-f_n(y)\|_{n+1} &\le \int_n^{n+1}\|T(n+1,r)(f(r,U(r,n)x)-f(r,U(r,n)y))\|_{n+1}\,dr\\
&\le 2\int_n^{n+1} e^{a(n+1-r)}\|f(r,U(r,n)x)-f(r,U(r,n)y)\|_r\,dr\\
&\le 2D''e^a\int_n^{n+1} p(r)\|U(r,n)x-U(r,n)y\|_r\,dr\\
&\le 4D''e^{2a}\|x-y\|_n e^{2D''\int_n^{n+1}p(r)\,dr}\int_n^{n+1}p(r)\,dr,
\end{aligned}$$

which implies that

$$\|f_n(x)-f_n(y)\| \le 4D''e^{2a}\mathcal{P}^{2D''}p_n\|x-y\|_n, \quad \text{for } n\in\mathbb{N} \text{ and } x,y\in X.$$

Denoting $A_n=T(n+1,n)$ we have that $A_n+f_n=U(n+1,n)$, $n\in\mathbb{N}$. Observe that A_n+f_n is invertible and $(A_n+f_n)^{-1}=U(n,n+1)$ for each $n\in\mathbb{N}$.

Let $\varepsilon>0$ be arbitrary and choose a continuously differentiable function $y\colon[0,\infty)\to X$ satisfying (3.16). Fix $n\in\mathbb{N}$ arbitrary and let $\bar{y}\colon[n,\infty)\to X$ and $z\colon[0,\infty)\to X$ be as in the proof of Theorem 2.7. We have (using (2.23), (3.16), (3.19) and (3.20)) that

$$\begin{aligned}\|y(t)-\bar{y}(t)\|_t &\le 2D''\int_n^t e^{a(t-r)}p(r)\|y(r)-\bar{y}(r)\|_r\,dr + 2\int_n^t e^{a(t-r)}\|z(r)\|_r\,dr\\ &\le 2e^aD''\varepsilon + 2e^aD''\int_n^t p(r)\|y(r)-\bar{y}(r)\|_r\,dr,\end{aligned}$$

for $n\in\mathbb{N}$ and $t\in[n,n+1]$. Hence, by applying Gronwall's lemma we conclude that

$$\|y(t)-\bar{y}(t)\|_t \le 2e^aD''\varepsilon e^{2e^aD''p_n} \quad \text{for } n\in\mathbb{N} \text{ and } t\in[n,n+1].$$

In particular,

$$\|y(n+1)-A_ny(n)-f_n(y(n))\|_{n+1} \le 2e^aD''\mathcal{P}^{2e^aD''}\varepsilon, \quad n\in\mathbb{N}.$$

Now, it follows from the proof of the previous theorem that there exists a sequence $(x_n)_{n\in\mathbb{N}}\subset X$ such that $x_{n+1}=A_nx_n+f_n(x_n)$, $n\in\mathbb{N}$ and

$$\|x_n-y(n)\|_n \le L\varepsilon \quad n\in\mathbb{N},$$

where $L>0$ depends only on D,D',λ,a and the sequence $(p_n)_n$. We define $x\colon[0,\infty)\to X$ by (2.25). Then, x satisfies (3.17) and (2.26) holds for $n\in\mathbb{N}$ and $t\in[n,n+1]$. Consequently,

$$\begin{aligned}\|x(t)-y(t)\|_t &\le 2e^a\|x_n-y(n)\|_n + 2e^aD''\varepsilon + 2e^aD''\int_n^t p(r)\|x(r)-y(r)\|_r\,dr\\ &\le (2e^aL+2e^aD'')\varepsilon + 2e^aD''\int_n^t p(r)\|x(r)-y(r)\|_r\,dr,\end{aligned}$$

for $t\in[n,n+1]$ and thus, by Gronwall's lemma, we get that

$$\|x(t)-y(t)\| \le \|x(t)-y(t)\|_t \le (2e^aL+2e^aD'')\mathcal{P}^{2e^aD''}\varepsilon, \quad t\ge 0.$$

Therefore, we conclude that (3.18) holds with $K:=(2e^aL+2e^aD'')\mathcal{P}^{2e^aD''}>0$ and the proof of the theorem is completed. □

Remark 3.8. Observe that condition (3.13) is fulfilled under some very natural assumptions. For instance, if $\nu(t)\ge 1$ for every $t\ge 0$ (which is the case we are in general interested in) then by Gronwall's lemma it follows easily that condition

$$\sup_{t\ge 0}\|A(t)\| < +\infty$$

implies that (3.13) is satisfied.

4. The dichotomic case

Based on the results obtained in the previous sections, one could wonder whether a similar result could be obtained in the case where Eq. (2.1) admits an exponential dichotomy. In this section, we show by means of an example that this is not the case in general. We will focus on the case of discrete-time dynamics. A similar example can be built in the continuous-time context. We start by recalling the following definition.

Definition 4.1. We say that (2.1) admits an *exponential dichotomy* if the following conditions are satisfied:

(1) there exists a family of projections P_n, $n \in \mathbb{N}$, such that
$$A_n P_n = P_{n+1} A_n; \tag{4.1}$$
(2) the restriction
$$A_n|_{\operatorname{Ker} P_n} \colon \operatorname{Ker} P_n \to \operatorname{Ker} P_{n+1} \tag{4.2}$$
is an invertible operator for each $n \in \mathbb{N}$;

(3) there exist $D, \lambda > 0$ such that
$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)} \quad \text{for } m \ge n \tag{4.3}$$
and
$$\|\mathcal{A}(m,n)(\operatorname{Id} - P_n)\| \le De^{-\lambda(n-m)} \quad \text{for } m \le n \tag{4.4}$$
where
$$\mathcal{A}(m,n) := \big(\mathcal{A}(n,m)|_{\operatorname{Ker} P_m}\big)^{-1} \colon \operatorname{Ker} P_n \to \operatorname{Ker} P_m,$$
for $m \le n$.

We have the following characterization of an exponential dichotomy which follows from [1, Corollary 2] and [1, Proposition 4] (see also [4]).

Theorem 4.2. *If $\dim(X) < +\infty$, then (2.1) admits an exponential dichotomy if and only if (2.1) is Lipschitz shadowable.*

We are now ready to construct our example.

Example 4.3. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of 2×2 matrices given by
$$A_n = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$$
for every $n \in \mathbb{N}$. It is easy to see that this sequence admits an exponential dichotomy with the sequence of projections $(P_n)_{n\in\mathbb{N}}$ given by
$$P_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Let us consider now the sequence $(B_n)_{n\in\mathbb{N}}$ given by
$$B_n = \begin{cases} A_n & \text{if } n \text{ is not a power of 2;} \\ B & \text{if } n \text{ is a power of 2} \end{cases}$$
where
$$B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$
Then, denoting by $\mathcal{B}(\cdot,\cdot)$ the evolution operator associated to $(B_n)_{n\in\mathbb{N}}$ and proceeding by induction we can show that, for every $n \ge 1$,
$$\mathcal{B}(2^n, 0) = \begin{pmatrix} 0 & e^{n-2} \\ 0 & e^{-(2^n-n)} \end{pmatrix}$$
and
$$\mathcal{B}(2^n + 1, 0) = \begin{pmatrix} 0 & e^{-(2^n-n)} \\ 0 & e^{-(2^n-n)} \end{pmatrix}.$$
In particular,
$$\lim_{n\to+\infty} \|\mathcal{B}(2^n + 1, 0)\| = 0$$
while
$$\lim_{n\to+\infty} \|\mathcal{B}(2^n, 0)\| = +\infty.$$

This shows that $(B_n)_{n\in\mathbb{N}}$ can not admit an exponential dichotomy. In particular, by Theorem 4.2, the difference equation

$$x_{n+1} = B_n x_n, \quad n \in \mathbb{N}$$

is not Lipschitz shadowable.

Let us now consider $f_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_n(x) = \begin{cases} 0 & \text{if } n \text{ is not a power of 2;} \\ (B - A_n)x & \text{if } n \text{ is a power of 2.} \end{cases}$$

Then, we have that

$$\|f_n(x) - f_n(y)\| \le p_n \|x - y\|$$

for every $x, y \in \mathbb{R}^2$ and $n \in \mathbb{N}$ where

$$p_n = \begin{cases} 0 & \text{if } n \text{ is not a power of 2;} \\ q & \text{if } n \text{ is a power of 2} \end{cases}$$

and

$$q := \left\| \begin{pmatrix} -e & 1 \\ 0 & 1 - e^{-1} \end{pmatrix} \right\|.$$

Then,

$$\lim_{n\to+\infty} \left(\prod_{j=0}^{n-1} (e^{-\lambda} + p_i) \right)^{\frac{1}{n}} < 1$$

and

$$\lim_{n\to+\infty} \left(\prod_{j=0}^{n-1} (e^{\lambda} - p_i) \right)^{\frac{1}{n}} > 1.$$

In particular, conditions similar to (2.4) and (3.3) are satisfied. Now, since (2.1) admits a *uniform* exponential dichotomy, a version of Theorems 2.4 and 3.2 in this context would have to give us that

$$x_{n+1} = A_n x + f_n(x), \quad n \in \mathbb{N}$$

is Lipschitz shadowable which we know is not the case since $A_n + f_n = B_n$, for every $n \in \mathbb{N}$. Consequently, a version of Theorems 2.4 and 3.2 for exponential dichotomic sequences does not hold in general.

Acknowledgments

We would like to thank the anonymous referee for his/hers useful comments that helped us improve and sharpen our results.

Declarations

Author contributions Both authors wrote the main manuscript text and contributed equally to the findings of the paper.

Data availability No datasets were generated or analysed during the current study.

Conflict of interest The authors declare no conflict of interest.

Funding L. Backes was partially supported by a CNPq-Brazil PQ fellowship under Grant No. 307633/2021-7. D. Dragičević was supported in part by University of Rijeka under the project uniri-iskusni-prirod-23-98 3046.

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Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, CEP 91509-900, Porto Alegre, RS, Brazil.

Email address, Lucas Backes: `lucas.backes@ufrgs.br`

Department of Mathematics, University of Rijeka, Croatia

Email address, Davor Dragičević: `ddragicevic@math.uniri.hr`