STABILITY OF NONAUTONOMOUS SYSTEMS ON FRÉCHET SPACES

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Abstract. The main objective of the present paper is to formulate sufficient conditions under which a nonautonomous dynamics acting on a arbitrary Fréchet space exhibits shadowing property and (partial) linearization. These conditions require that the linear part is hyperbolic (in the sense of the concept recently introduced by Aragão Costa) and that the nonlinear part is Lipschitz. Our results extend those previously known in the setting of nonautonomous dynamics acting on Banach space. We consider both the case of discrete and continuous time dynamics.

1. Introduction

In general, a dynamical system is said to be *stable* if certain dynamical properties are preserved under small perturbations of a system. Clearly, we have several notions of stability depending, for instance, on which dynamical properties we want to be preserved and how do we measure the smallness of the perturbations. In the present work we are interested in two such notions: shadowing and linearization. More precisely, given a dynamical system of the form

$$x_{n+1} = A_n x_n + f_n(x_n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $A_n \colon X \to X$, $n \in \mathbb{Z}$, is sequence of linear maps acting on a Fréchet space X and $f_n \colon X \to X$, $n \in \mathbb{Z}$, is a sequence of nonlinear maps, we are interested in describing sufficient conditions under which the system (1.1) and its continuous time counterpart have the shadowing property and are linearizable.

The shadowing theory deals with the problem of finding exact trajectories of a dynamical system close to *pseudo-orbits*. In the case of the dynamical system generated by (1.1), for instance, a pseudo-orbit is a sequence of points $(y_n)_{n\in\mathbb{Z}}$ in X which is almost an orbit of the system: for each $n \in \mathbb{Z}$, the difference $y_{n+1} - A_n y_n - f_n(y_n)$ does not need to be zero (as it is in the case of an actual orbit) but must, on the other hand, be small. We say that a dynamical system exhibits the *shadowing property* if close to a pseudo-orbit we can find an exact orbit. This theory has a long history starting with the seminal work of Bowen [10, 11]. Afterwards, many authors have given several interesting and deep contributions to broadening its scope of applicability and also obtaining important dynamical consequences arising from shadowing. We refer to the excellent monographs of Palmer [25], Pilyugin [26] and Pilyugin-Sakai [28] for a thorough overview of these results. We just mention that initially the theory was developed for autonomous systems defined on compact spaces and, more recently, was extended to both autonomous and nonautonomous context for infinite dimensional Banach spaces. See, for instance, [2, 3, 6, 8, 9, 12,

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13, 27] and the references therein for an overview of these later developments. In this work we go one step further and study the shadowing property for nonautonomous dynamics acting on Frećhet spaces. In the sequel we explain the importance of considering this setting but first let us turn our attention to the other stability property that we are going to explore.

We start by recalling that the system (1.1) is said to be *linearizable* if there exists a sequence of homeomorphisms $H_n \colon X \to X$ sending the trajectories of (1.1) into the trajectories of the linear system

$$x_{n+1} = A_n x_n, \; n \in \mathbb{Z}. \tag{1.2}$$

In particular, whenever this happens, many important dynamical properties of the nonlinear system (1.1) can be obtained by studying the linear system (1.2), which in general is much easier to deal with. Although this is a nice property, it is also a very strong one and, in particular, may fail to be true in many interesting situations. A weaker version of this property is what we call *partial linearization*: instead of looking for a transformation that sends *all* the trajectories of (1.1) to trajectories of (1.2), we restrict ourselves to trajectories starting in some special invariant subspaces of (1.2). This notion will be made more precise when we are going to state our results in Section 4. Probably the first and most well-known results regarding linearization are due to Grobman [18, 19] and Hartman [20, 21, 22] in the context of autonomous dynamics and by Palmer [24] in the context of nonautonomous dynamics. Following the lead of these precursors, many authors have worked to expand the reach of applicability of these type of results by considering, for instance, more general dynamics acting on very general phase spaces, by weakening the notions of linearization and also by improving the regularity of the conjugacy maps H_n. We refer to the introductions of [4, 5, 7, 14, 15, 16, 23] for a more thorough revision of the literature regarding this subject.

As already mentioned, our main objective in this work is to present sufficient conditions under which a system of the form (1.1) together with its continuous time counterpart have the shadowing property and are (partially) linearizable. Our major assumption is that the linear system (1.2) admits a type of exponential dichotomy (introduced by Aragão Costa [1]) and that the nonlinear perturbations f_n, $n \in \mathbb{Z}$, are Lipschitz with small Lipschitz constant. These are somehow classical assumptions when dealing with this kind of problems but the main novelty of this work with respect to previous ones is that here we work on Fréchet spaces. This level of generality allows us to apply our results, for instance, to dynamics defined via differential operators. This type of linear operators, when thought as operators acting on a Banach space, are, in general, not bounded and, in particular, previously available results can not be applied to them. On the other hand, they may fit into our framework as observed in Example 3. This example deals with the Laplace operator acting on the space of tempered distributions and satisfies, for instance, the hypothesis of Theorems 1 and 3. In particular, small perturbations of it have the shadowing property and are partially linearizable.

The paper is organized as follows: in Section 2 we recall some notions regarding Fréchet spaces and introduce the notions of exponential dichotomy in this setting for both discrete and continuous time dynamical systems. Moreover, we present some examples of systems satisfying this property. In Sections 3 and 4 we present sufficient conditions under which the system (1.1) and its continuous time counterpart have the shadowing property and are (partially) linearizable, respectively.

2. Preliminaries

In this section we recall some important notions that are going to be used in the paper.

2.1. **Fréchet spaces.** A complete Hausdorff topological vector space whose topology is generated by a countable family of seminorms is called a *Fréchet space*. Throughout this paper X will denote an arbitrary Fréchet space whose topology is generated by a sequence $(p_j)_{j\in\mathbb{N}}$ of seminorms on X. Then, we have (see [17, 5.16 Proposition]) that

$$x = 0 \text{ in } X \iff p_j(x) = 0 \text{ for all } j \in \mathbb{N}. \tag{2.1}$$

We recall that $B \subset X$ is said to be bounded if for every $j \in \mathbb{N}$, the set $\{p_j(x) : x \in B\} \subset \mathbb{R}$ is bounded. By $\mathcal{L}(X)$, we will denote the set of all linear operators $A\colon X \to X$ with the property that $A(B)$ is a bounded set for each bounded $B \subset X$. One can easily check (see [17]) that if $A\colon X \to X$ is a linear operator then the following conditions are equivalent:

- $A \in \mathcal{L}(X)$;
- A is continuous;
- for each $j \in \mathbb{N}$ there exists $k_j \in \mathbb{N}$ and $C_j > 0$ such that

$$p_j(Ax) \leqslant C_j \sum_{i=1}^{k_j} p_i(x) \text{ for every } x \in X.$$

Example 1. Every Banach space $(X, \|\cdot\|)$ is a Fréchet space with $p_j(\cdot) = \|\cdot\|$, $j \in \mathbb{N}$, and in this case we obviously have that $\mathcal{L}(X)$ is just the space of bounded linear operators on $(X, \|\cdot\|)$.

Example 2. Let $X = C^\infty([0,1])$ be the space of all infinitely differentiable functions $f\colon [0,1] \to \mathbb{R}$ and, for each $j \in \mathbb{N}$, consider the seminorm

$$p_j(f) = \sup\left\{\left|\frac{d^j}{dx^j} f(x)\right| : \ x \in [0,1]\right\}.$$

Then, it is not difficult to check that $(X, (p_j)_{j\in\mathbb{N}})$ is a Fréchet space and $\frac{d}{dx} \in \mathcal{L}(X)$. On the other hand, there is no norm in X with respect to which $\frac{d}{dx}$ is bounded. In particular, this example show us that the notion of Fréchet space is an important generalization of the notion of Banach space.

2.2. **Exponential dichotomy.** In this section we present versions of the notion of exponential dichotomy in the setting of Fréchet spaces.

2.2.1. *The discrete time case.* For a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$, we denote by $\mathcal{A}(m,n)$ the associated linear cocycle defined by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & m > n; \\ \mathrm{Id} & m = n, \end{cases}$$

where Id denotes the identity operator on X. We now recall the notion of exponential dichotomy introduced in [1, Definition 2.13.].

Definition 1. *We say that a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ admits an* exponential dichotomy *if there exist two sequences $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ of positive numbers, as well as a sequence $(Q_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ of projections such that the following conditions hold:*

- *for* $n \in \mathbb{Z}$,
$$A_n Q_n = Q_{n+1} A_n;$$
- *for every* $n \in \mathbb{Z}$, $A_n|_{R(Q_n)} \colon R(Q_n) \to R(Q_{n+1})$ *is an isomorphism, where* $R(Q_n)$ *denotes the range of* Q_n*;*
- *for* $m \geqslant n$, $j \in \mathbb{N}$ *and* $x \in X$,
$$p_j(\mathcal{A}(m,n)(\mathrm{Id} - Q_n)x) \leqslant M_j e^{-w_j(m-n)} p_j(x); \tag{2.2}$$
- *for* $m \leqslant n$, $j \in \mathbb{N}$ *and* $x \in X$,
$$p_j(\mathcal{A}(m,n) Q_n x) \leqslant M_j e^{-w_j(n-m)} p_j(x), \tag{2.3}$$
where
$$\mathcal{A}(m,n) := \left(\mathcal{A}(n,m)|_{R(Q_m)}\right)^{-1} : R(Q_n) \to R(Q_m), \quad m < n.$$

The following interesting example is taken from [1, Example 3.5].

Example 3. Let us denote by $\mathcal{S}(\mathbb{R}^n)$ the space of *Schwartz functions* or *rapidly decreasing functions* $f \colon \mathbb{R}^n \to \mathbb{C}$ (see [17]). Then, a *tempered distribution* on $\mathbb{R}^n$ is just a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}'(\mathbb{R}^n)$ denote the set of all tempered distributions on $\mathbb{R}^n$ and Ω be an open subset of $\mathbb{R}^n$. On $\mathcal{S}'(\mathbb{R}^n)$ we consider the equivalence relation given by
$$u \sim_\Omega v \iff \hat{u}|_\Omega = \hat{v}|_\Omega,$$
where $\hat{g}$ denotes the Fourier transform of $g \in \mathcal{S}'(\mathbb{R}^n)$.

Let u_Ω denote the equivalence class of $u \in \mathcal{S}'(\mathbb{R}^n)$ with respect to '$\sim_\Omega$' and consider
$$E_\Omega = \{u_\Omega : \ \hat{u}|_\Omega \in L^2_{\mathrm{loc}}(\Omega) \text{ for some } u \in u_\Omega\}.$$
Then, E_Ω is a vector space. Now, given a sequence $(K_j)_{j\in\mathbb{N}}$ of compact sets such that $K_j \subset K_{j+1}$ for every $j \in \mathbb{N}$ and $\Omega = \bigcup_{j\in\mathbb{N}} K_j$, let us consider the sequence of seminorms on E_Ω given by
$$p_j^*(u_\Omega) = \left(\int_{K_j} |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2} \tag{2.4}$$
for every $j \in \mathbb{N}$, $u \in u_\Omega$ and $u_\Omega \in E_\Omega$. It is not difficult to check that $(p_j^*)_{j\in\mathbb{N}}$ is indeed a separating sequence of seminorms on E_Ω and, consequently,
$$d(u_\Omega, v_\Omega) = \sum_{j=1}^{+\infty} \frac{p_j^*(u_\Omega - v_\Omega)}{2^j(1 + p_j^*(u_\Omega - v_\Omega))}$$
is a metric on E_Ω.

Let us consider now $\mathcal{F}L^2_{\mathrm{loc}}(\Omega)$ the completion of the metric space (E_Ω, d). Here we think of the completion as the quotient space of all Cauchy sequences in E_Ω by the equivalence relation given by
$$(u_{\Omega,l})_{l\in\mathbb{N}} \sim (v_{\Omega,l})_{l\in\mathbb{N}} \iff \lim_{l\to+\infty} d(u_{\Omega,l}, v_{\Omega,l}) = 0.$$
One can check that the seminorms given in (2.4) have natural extensions to $\mathcal{F}L^2_{\mathrm{loc}}(\Omega)$ and that this space endowed with these seminorms is a Fréchet space.

Given $[u] \in \mathcal{F}L^2_{\mathrm{loc}}(\Omega)$, we may define its Fourier transform as follows: for any $(u_{\Omega,l})_{l\in\mathbb{N}} \in [u]$ we have that $(\hat{u}_{\Omega,l})_{l\in\mathbb{N}}$ is a Cauchy sequence in $L^2_{\mathrm{loc}}(\Omega)$ and, therefore, there exists a unique $w \in L^2_{\mathrm{loc}}(\Omega)$ such that $\hat{u}_{\Omega,l} \to w$ in $L^2_{\mathrm{loc}}(\Omega)$. We then define

$$\widehat{[u]} = w.$$

It is easy to check that this definition does not depend on the representative $(u_{\Omega,l})_{l\in\mathbb{N}}$ of $[u]$ and, moreover, when restricted to elements of E_Ω this Fourier transform coincide with the ordinary one in E_Ω. Finally, if $[u] \in \mathcal{F}L^2_{\mathrm{loc}}(\Omega)$ and $\widehat{[u]} = w$ then we define $\check{w} = [u]$ as the inverse Fourier transform, which is well defined as one can easily check.

Let us now specialize our example to the case when $\Omega = \mathbb{R}^n\backslash\{0\}$. Denoting by $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ the Laplace operator on $\mathbb{R}^n\backslash\{0\}$, let us consider the operator $T\colon E_{\mathbb{R}^n\backslash\{0\}} \to E_{\mathbb{R}^n\backslash\{0\}}$ given by

$$Tu = e^{\Delta}u := \left(e^{-4\pi^2|\xi|^2}\hat{u}\right)\check{}.$$

This operator has a natural extension to $\mathcal{F}L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ given by $T\colon \mathcal{F}L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) \to \mathcal{F}L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$,

$$T[u] := [(T(u_{\mathbb{R}^n\backslash\{0\},l}))_{l\in\mathbb{N}}]$$

for $(u_{\mathbb{R}^n\backslash\{0\},l})_{l\in\mathbb{N}} \in [u] \in \mathcal{F}L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$. In this case we have that

$$\widehat{T[u]} = e^{-4\pi^2|\xi|^2}w$$

where $w = \widehat{[u]} \in L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$.

Considering now $K_j = \{x \in \mathbb{R}^n : 1/j \leqslant |x| \leqslant j\}$ we have that

$$p_j^*(T[u])^2 = \int_{1/j\leqslant|\xi|\leqslant j} |e^{-4\pi^2|\xi|^2}w(\xi)|^2\,d\xi.$$

Thus, for every $n \in \mathbb{N}$ it follows that

$$p_j^*(T^n[u]) = \left(\int_{1/j\leqslant|\xi|\leqslant j} |e^{-4n\pi^2|\xi|^2}w(\xi)|^2\,d\xi\right) \leqslant e^{-4n\pi^2/j^2}p_j^*([u]).$$

Consequently, the sequence $(A_n)_{n\in\mathbb{Z}} = (T)_{n\in\mathbb{Z}} = (e^{\Delta})_{n\in\mathbb{Z}}$ admits an exponential dichotomy with constants $M_j = 1$ and $\omega_j = 4\pi^2/j^2$ for every $j \in \mathbb{N}$. On the other hand, we recall that, as observed in [1, Introduction], $(A_n)_{n\in\mathbb{Z}} = (e^{\Delta})_{n\in\mathbb{Z}}$ does not admit an exponential dichotomy when thought of as acting, for instance, in the Banach space $L^2(\mathbb{R}^n)$.

2.2.2. *The continuous time case.*

Definition 2. *Let $T(t,s)$, $t \geqslant s$ be a two parameter family of linear operators on X. We say that $T(t,s)$ is an* evolution family *if the following properties hold:*

- $T(t,t) = \mathrm{Id}$ *for* $t \in \mathbb{R}$*;*
- *for* $t \geqslant s \geqslant r$*,*
$$T(t,s)T(s,r) = T(t,r);$$
- *for* $(s,x) \in \mathbb{R} \times X$*,* $t \mapsto T(t,s)x$ *is continuous on* $[s,\infty)$*;*
- *for each* $j \in \mathbb{N}$*, there exist* $K_j, a_j > 0$ *such that*
$$p_j(T(t,s)x) \leqslant K_j e^{a_j(t-s)}p_j(x), \quad \text{for } t \geqslant s. \tag{2.5}$$

Remark 1. Observe that (2.5) implies that $T(t,s) \in \mathcal{L}(X)$ for $t \geqslant s$.

We now introduce the notion of exponential dichotomy for evolution families.

Definition 3. *We say that an evolution family $T(t,s)$ admits an* exponential dichotomy *if there exist two sequences $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ of positive numbers, as well as a family $Q(s)\in\mathcal{L}(X)$, $s\in\mathbb{R}$ of projections such that the following conditions hold:*

- *for $t\geqslant s$,*
$$T(t,s)Q(s)=Q(t)T(t,s);$$
- *for $t\geqslant s$, $T(t,s)|_{R(Q(s))}\colon R(Q(s))\to R(Q(t))$ is an isomorphism;*
- *for $t\geqslant s$, $j\in\mathbb{N}$ and $x\in X$,*
$$p_j(T(t,s)(\mathrm{Id}-Q(s))x)\leqslant M_je^{-w_j(t-s)}p_j(x); \tag{2.6}$$
- *for $t\leqslant s$, $j\in\mathbb{N}$ and $x\in X$,*
$$p_j(T(t,s)Q(s)x)\leqslant M_je^{-w_j(s-t)}p_j(x), \tag{2.7}$$
where $T(t,s)$ for $t<s$ denotes the inverse of $T(s,t)|_{R(Q(t))}$.

As in the discrete time case, it is not difficult to present examples of evolution families admitting an exponential dichotomy. For instance, we can consider the continuous time version of Example 3 given by $T(t,s)=e^{(t-s)\Delta}$, $t\geqslant s$. Now, proceeding as in the above mentioned example one can check that $T(t,s)$ is actually an evolution family that admits an exponential dichotomy. An even simpler example is the following.

Example 4. Consider the 'annulus' $\mathbb{A}=\{x\in\mathbb{R}^n;\ |x|\geqslant 1/2\}$ and $L^1_{\mathrm{loc}}(\mathbb{A})$ endowed with the sequence of seminorms

$$p_j(f)=\int_{\{\xi\in\mathbb{A};\ |\xi|\leqslant j\}}|f(\xi)|d\xi,\ j\in\mathbb{N}.$$

Then, $(L^1_{\mathrm{loc}}(\mathbb{A}),(p_j)_{j\in\mathbb{N}})$ is a Fréchet space. Set $X:=L^1_{\mathrm{loc}}(\mathbb{A})\times L^1_{\mathrm{loc}}(\mathbb{A})$ and

$$p_j^X((f,g)):=\max\{p_j(f),p_j(g)\},\quad j\in\mathbb{N},\ (f,g)\in X.$$

Then, $(X,(p_j^X)_{j\in\mathbb{N}})$ is also a Fréchet space. For $t\geqslant s$, we define $T(t,s)\colon X\to X$ by

$$T(t,s)(f,g)(\xi_1,\xi_2)=(e^{-(t-s)|\xi_1|}f(\xi_1),e^{(t-s)|\xi_2|}g(\xi_2)),$$

for $(\xi_1,\xi_2)\in\mathbb{A}\times\mathbb{A}$ and $(f,g)\in X$. Moreover, let $Q(s)\colon L^1_{\mathrm{loc}}(\mathbb{A})\times L^1_{\mathrm{loc}}(\mathbb{A})\to L^1_{\mathrm{loc}}(\mathbb{A})\times L^1_{\mathrm{loc}}(\mathbb{A})$ be given by $Q(s)(f,g)=(0,g)$ for every $s\in\mathbb{R}$. Then, it is not difficult to see that $T(t,s)$ is an evolution family that admits an exponential dichotomy with family of projections $Q(s)$, $s\in\mathbb{R}$.

3. The shadowing property

In this section we are going to present sufficient conditions under which a certain type of dynamical systems defined on a Fréchet space exhibit the shadowing property.

3.1. **The discrete time case.** In what follows, for a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ we will consider the associated linear nonautonomous dynamics given by

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{3.1}$$

Furthermore, let $f_n \colon X \to X$, $n \in \mathbb{Z}$ be a sequence of (nonlinear) maps. We consider the nonlinear nonautonomous dynamics given by

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z}. \tag{3.2}$$

The following is our first result.

Theorem 1. *Assume that a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ admits an exponential dichotomy with sequences of constants $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$. Let $(c_j)_{j\in\mathbb{N}} \subset [0,\infty)$ be such that*

$$c_j M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} < 1, \quad \text{for } j \in \mathbb{N}. \tag{3.3}$$

Moreover, set

$$C_j := \frac{M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}}{1 - c_j M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}} > 0, \quad j \in \mathbb{N}.$$

Finally, let $f_n \colon X \to X$, $n \in \mathbb{Z}$ be a sequence of maps with the property that

$$p_j(f_n(x) - f_n(y)) \leqslant c_j p_j(x-y), \quad \text{for } n \in \mathbb{Z},\ j \in \mathbb{N} \text{ and } x, y \in X. \tag{3.4}$$

Then, for every two sequences $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ and $(y_n)_{n\in\mathbb{Z}} \subset X$ such that

$$p_j(y_{n+1} - A_n y_n - f_n(y_n)) \leqslant \varepsilon_j \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N}, \tag{3.5}$$

there exists a unique sequence $(x_n)_{n\in\mathbb{Z}} \subset X$ satisfying (3.2) and

$$p_j(x_n - y_n) \leqslant C_j \varepsilon_j, \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N}. \tag{3.6}$$

Remark 2. In other words, what our first result is saying is that for any *pseudo-orbit* $(y_n)_{n\in\mathbb{Z}}$ of (3.2) there exists an actual solution of (3.2) that *shadows* $(y_n)_{n\in\mathbb{Z}}$ (condition (3.6)). In particular, the system (3.2) has the so-called *shadowing property* (see [25, 26]).

Proof. Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ and $(y_n)_{n\in\mathbb{Z}} \subset X$ be as in the statement of the theorem. We define a sequence $\mathbf{z}^1 = (z_n^1)_{n\in\mathbb{Z}} \subset X$ by

$$\begin{aligned} z_n^1 = & \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k) \\ & - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k) Q_k (A_{k-1}y_{k-1} + f_{k-1}(y_{k-1}) - y_k), \end{aligned}$$

for $n \in \mathbb{Z}$. By (2.2), (2.3) and (3.5), we have that

$$\begin{aligned}
p_j(z_n^1) &\leqslant \sum_{k=-\infty}^{n} p_j(\mathcal{A}(n,k)(\text{Id}-Q_k)(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k)) \\
&\quad + \sum_{k=n+1}^{\infty} p_j(\mathcal{A}(n,k)Q_k(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k)) \\
&\leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)} p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k) \\
&\quad + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)} p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k) \\
&\leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)}\varepsilon_j + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)}\varepsilon_j \\
&= M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}\varepsilon_j,
\end{aligned}$$

and thus

$$p_j(z_n^1) \leqslant M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}\varepsilon_j, \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N}. \tag{3.7}$$

Next, we define a sequence $\mathbf{z}^2 = (z_n^2)_{n\in\mathbb{Z}} \subset X$ by

$$\begin{aligned}
z_n^2 &= \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\text{Id}-Q_k)(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1}^1)-y_k) \\
&\quad - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1}^1)-y_k),
\end{aligned}$$

for $n \in \mathbb{Z}$. Observe that (3.4) and (3.7) imply that

$$\begin{aligned}
&p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1}^1)-y_k) \\
&\leqslant p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k) \\
&\quad + p_j(f_{k-1}(y_{k-1}+z_{k-1}^1)-f_{k-1}(y_{k-1})) \\
&\leqslant p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k)+c_j p_j(z_{k-1}^1) \\
&\leqslant \varepsilon_j + c_j M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}\varepsilon_j,
\end{aligned}$$

for $k \in \mathbb{Z}$ and $j \in \mathbb{N}$. Hence, using (2.2) and (2.3) we obtain that

$$p_j(z_n^2) \leqslant M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}\varepsilon_j \left(1 + c_j M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}\right),$$

for every $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. Thus, the sequence $\mathbf{z}^2$ is well-defined. On the other hand, observe that

$$\begin{aligned}
z_n^2 - z_n^1 &= \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\text{Id}-Q_k)(f_{k-1}(y_{k-1}+z_{k-1}^1)-f_{k-1}(y_{k-1})) \\
&\quad - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(y_{k-1}+z_{k-1}^1)-f_{k-1}(y_{k-1})),
\end{aligned}$$

and consequently (using (2.2), (2.3), (3.4) and (3.7))

$$p_j(z_n^2 - z_n^1) \leqslant c_j\left(M_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^2\varepsilon_j, \quad \text{for } n\in\mathbb{Z} \text{ and } j\in\mathbb{N}.$$

We proceed inductively: suppose that we have constructed $\mathbf{z}^l = (z_n^l)_{n\in\mathbb{Z}} \subset X$ such that

$$p_j(z_n^l - z_n^{l-1}) \leqslant c_j^{l-1}\left(M_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^l\varepsilon_j \quad \text{for } n\in\mathbb{Z} \text{ and } j\in\mathbb{N}, \tag{3.8}$$

and

$$p_j(z_n^l) \leqslant \varepsilon_j M_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\sum_{i=0}^{l-1}\left(c_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^i, \tag{3.9}$$

for $n\in\mathbb{Z}$ and $j\in\mathbb{N}$. We define a sequence $\mathbf{z}^{l+1} = (z_n^{l+1})_{n\in\mathbb{Z}} \subset X$ by

$$\begin{aligned} z_n^{l+1} = &\sum_{k=-\infty}^{n}\mathcal{A}(n,k)(\mathrm{Id}-Q_k)(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1}^l)-y_k) \\ &-\sum_{k=n+1}^{\infty}\mathcal{A}(n,k)Q_k(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1}^l)-y_k), \end{aligned} \tag{3.10}$$

for $n\in\mathbb{Z}$. Observe that (3.4) and (3.9) imply that

$$\begin{aligned} &p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+z_{k-1}^l)-y_k) \\ &\leqslant p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k) \\ &\quad + p_j(f_{k-1}(y_{k-1}+z_{k-1}^l)-f_{k-1}(y_{k-1})) \\ &\leqslant \varepsilon_j + c_jp_j(z_{k-1}^l) \\ &\leqslant \varepsilon_j + c_j\varepsilon_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\sum_{i=0}^{l-1}\left(c_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^i, \end{aligned}$$

for $k\in\mathbb{Z}$ and $j\in\mathbb{N}$. Hence, (2.2) and (2.3) give that

$$\begin{aligned} p_j(z_n^{l+1}) &\leqslant M_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\left(\varepsilon_j + c_j\varepsilon_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\sum_{i=0}^{l-1}\left(c_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^i\right) \\ &= \varepsilon_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\sum_{i=0}^{l}\left(c_jM_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^i, \end{aligned}$$

for $n\in\mathbb{Z}$ and $j\in\mathbb{N}$. In particular, the sequence $\mathbf{z}^{l+1}$ is well-defined. Moreover,

$$\begin{aligned} z_n^{l+1} - z_n^l = &\sum_{k=-\infty}^{n}\mathcal{A}(n,k)(\mathrm{Id}-Q_k)(f_{k-1}(y_{k-1}+z_{k-1}^l)-f_{k-1}(y_{k-1}+z_{k-1}^{l-1})) \\ &-\sum_{k=n+1}^{\infty}\mathcal{A}(n,k)Q_k(f_{k-1}(y_{k-1}+z_{k-1}^l)-f_{k-1}(y_{k-1}+z_{k-1}^{l-1})). \end{aligned}$$

Hence, (2.2), (2.3), (3.4) and (3.8) imply that

$$p_j(z_n^{l+1} - z_n^l) \leqslant c_j^l\left(M_j\frac{1+e^{-w_j}}{1-e^{-w_j}}\right)^{l+1}\varepsilon_j, \quad \text{for } n\in\mathbb{Z} \text{ and } j\in\mathbb{N}. \tag{3.11}$$

Thus, we have constructed a sequence $(\mathbf{z}^l)_{l\in\mathbb{N}}$, $\mathbf{z}^l = (z_n^l)_{n\in\mathbb{Z}} \subset X$ satisfying (3.11) and

$$p_j(z_n^l) \leqslant C_j\varepsilon_j, \quad \text{for } n \in \mathbb{Z} \text{ and } j, l \in \mathbb{N}. \tag{3.12}$$

By (3.3) and (3.11), we conclude that $(z_n^l)_{l\in\mathbb{N}}$ is a Cauchy sequence in X for each $n \in \mathbb{Z}$. Let

$$z_n := \lim_{l\to\infty} z_n^l, \quad n \in \mathbb{Z}.$$

It follows from (3.12) that

$$p_j(z_n) \leqslant C_j\varepsilon_j, \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N}. \tag{3.13}$$

Moreover, (3.10) implies that

$$\begin{aligned} z_n = &\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k) \\ &- \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k), \end{aligned} \tag{3.14}$$

for $n \in \mathbb{Z}$. Then,

$$\begin{aligned} &z_{n+1} - A_n z_n \\ &= \sum_{k=-\infty}^{n+1} \mathcal{A}(n+1,k)(\mathrm{Id} - Q_k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k) \\ &\quad - \sum_{k=-\infty}^{n} \mathcal{A}(n+1,k)(\mathrm{Id} - Q_k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k) \\ &\quad - \sum_{k=n+2}^{\infty} \mathcal{A}(n+1,k)Q_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k) \\ &\quad + \sum_{k=n+1}^{\infty} \mathcal{A}(n+1,k)Q_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - y_k) \\ &= (\mathrm{Id} - Q_{n+1})(A_n y_n + f_n(y_n + z_n) - y_{n+1}) \\ &\quad + Q_{n+1}(A_n y_n + f_n(y_n + z_n) - y_{n+1}) \\ &= A_n y_n + f_n(y_n + z_n) - y_{n+1}, \end{aligned} \tag{3.15}$$

for each $n \in \mathbb{Z}$. Setting

$$x_n := y_n + z_n \quad n \in \mathbb{Z},$$

we conclude readily from (3.15) that (3.2) holds. Moreover, (3.13) implies (3.6).

Finally, let $(\tilde{x}_n)_{n\in\mathbb{Z}} \subset X$ be another sequence such that

$$\tilde{x}_{n+1} = A_n\tilde{x}_n + f_n(\tilde{x}_n) \quad n \in \mathbb{Z}, \tag{3.16}$$

and

$$p_j(\tilde{x}_n - y_n) \leqslant C_j\varepsilon_j, \quad \text{for } j \in \mathbb{N} \text{ and } n \in \mathbb{Z}.$$

Let

$$\tilde{z}_n := \tilde{x}_n - y_n, \quad n \in \mathbb{Z}. \tag{3.17}$$

Thus,

$$p_j(\tilde{z}_n) \leqslant C_j\varepsilon_j, \quad \text{for } j \in \mathbb{N} \text{ and } n \in \mathbb{Z}. \tag{3.18}$$

Next, for $n \in \mathbb{Z}$ set

$$\begin{aligned}\tilde{z}_n' := &\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+\tilde{z}_{k-1})-y_k)\\ &-\sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+\tilde{z}_{k-1})-y_k).\end{aligned}$$

Observe that

$$\begin{aligned}&p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1}+\tilde{z}_{k-1})-y_k)\\ &\leqslant p_j(A_{k-1}y_{k-1}+f_{k-1}(y_{k-1})-y_k)\\ &\quad+p_j(f_{k-1}(y_{k-1}+\tilde{z}_{k-1})-f_{k-1}(y_{k-1}))\\ &\leqslant \varepsilon_j+c_jC_j\varepsilon_j\\ &=(1+c_jC_j)\varepsilon_j,\end{aligned}$$

for $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. This together with (2.2) and (2.3) gives that

$$p_j(\tilde{z}_n') \leqslant M_j \frac{1+e^{-w_j}}{1-e^{-w_j}}(1+c_jC_j)\varepsilon_j, \quad \text{for } j \in \mathbb{N} \text{ and } n \in \mathbb{Z}. \tag{3.19}$$

On the other hand, we have (see (3.16) and (3.17)) that

$$\begin{aligned}\tilde{z}_n' &= \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(A_{k-1}\tilde{x}_{k-1}-A_{k-1}\tilde{z}_{k-1}+f_{k-1}(\tilde{x}_{k-1})-y_k)\\ &\quad-\sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(A_{k-1}\tilde{x}_{k-1}-A_{k-1}\tilde{z}_{k-1}+f_{k-1}(\tilde{x}_{k-1})-y_k)\\ &=\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(\tilde{z}_k-A_{k-1}\tilde{z}_{k-1})\\ &\quad-\sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(\tilde{z}_k-A_{k-1}\tilde{z}_{k-1}),\end{aligned}$$

for $n \in \mathbb{Z}$. Therefore,

$$\begin{aligned}\tilde{z}_{n+1}'-A_n\tilde{z}_n' &= \sum_{k=-\infty}^{n+1} \mathcal{A}(n+1,k)(\mathrm{Id}-Q_k)(\tilde{z}_k-A_{k-1}\tilde{z}_{k-1})\\ &\quad-\sum_{k=-\infty}^{n} \mathcal{A}(n+1,k)(\mathrm{Id}-Q_k)(\tilde{z}_k-A_{k-1}\tilde{z}_{k-1})\\ &\quad-\sum_{k=n+2}^{\infty} \mathcal{A}(n+1,k)Q_k(\tilde{z}_k-A_{k-1}\tilde{z}_{k-1})\\ &\quad+\sum_{k=n+1}^{\infty} \mathcal{A}(n+1,k)Q_k(\tilde{z}_k-A_{k-1}\tilde{z}_{k-1})\\ &=(\mathrm{Id}-Q_{n+1})(\tilde{z}_{n+1}-A_n\tilde{z}_n)+Q_{n+1}(\tilde{z}_{n+1}-A_n\tilde{z}_n)\\ &=\tilde{z}_{n+1}-A_n\tilde{z}_n,\end{aligned}$$

and thus

$$\tilde{z}_{n+1}'-\tilde{z}_{n+1}=A_n(\tilde{z}_n'-\tilde{z}_n), \quad n \in \mathbb{Z}. \tag{3.20}$$

Set $r_n := \tilde{z}_n' - \tilde{z}_n$, $n \in \mathbb{Z}$. By (3.20), we have that $r_{n+1} = A_n r_n$ for $n \in \mathbb{Z}$. Moreover, (3.18) and (3.19) imply that for each $j \in \mathbb{N}$, there exists $D_j > 0$ such that

$$p_j(r_n) \leqslant D_j, \quad n \in \mathbb{Z}.$$

Fix now an arbitrary $j \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then, for each $m \geqslant 0$ we have (see (2.2)) that

$$p_j((\mathrm{Id} - Q_n)r_n) = p_j(\mathcal{A}(n, n-m)(\mathrm{Id} - Q_{n-m})r_{n-m}) \leqslant M_j D_j e^{-w_j m}.$$

Letting $m \to \infty$, we conclude that $p_j((\mathrm{Id} - Q_n)r_n) = 0$. Since $j \in \mathbb{N}$ was arbitrary, from (2.1) we have that $(\mathrm{Id} - Q_n)r_n = 0$ for each $n \in \mathbb{Z}$. Similarly, using (2.3) it follows that $Q_n r_n = 0$ for each $n \in \mathbb{Z}$. We conclude that $r_n = 0$, and thus $\tilde{z}_n' = \tilde{z}_n$ for every $n \in \mathbb{Z}$. Hence,

$$\begin{aligned}\tilde{z}_n := &\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + \tilde{z}_{k-1}) - y_k) \\ &- \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(A_{k-1}y_{k-1} + f_{k-1}(y_{k-1} + \tilde{z}_{k-1}) - y_k),\end{aligned} \tag{3.21}$$

for $n \in \mathbb{Z}$. By (2.2), (2.3), (3.4), (3.14) and (3.21), we have that

$$\sup_n p_j(\tilde{z}_n - z_n) \leqslant M_j c_j \frac{1 + e^{-w_j}}{1 - e^{-w_j}} \sup_n p_j(\tilde{z}_n - z_n),$$

for each $j \in \mathbb{N}$. Thus, it follows from (3.3) that $p_j(\tilde{z}_n - z_n) = 0$ for every $j \in \mathbb{N}$ and $n \in \mathbb{Z}$. Hence, (2.1) implies that $\tilde{z}_n = z_n$ for $n \in \mathbb{Z}$. We conclude that $\tilde{x}_n = x_n$ for every $n \in \mathbb{Z}$. The proof of the theorem is completed. □

Remark 3. In the case when X is a Banach space, Theorem 1 follows from [3, Theorem 3].

We now emphasize some important special cases of Theorem 1.

Corollary 1. *Assume that a sequence $(A_n)_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ admits an exponential dichotomy with sequences of constants $(M_j)_{j \in \mathbb{N}}$ and $(w_j)_{j \in \mathbb{N}}$. Moreover, let $(z_n)_{n \in \mathbb{Z}} \subset X$ be an arbitrary sequence. Then, for every two sequences $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, \infty)$ and $(y_n)_{n \in \mathbb{Z}} \subset X$ such that*

$$p_j(y_{n+1} - A_n y_n - z_n) \leqslant \varepsilon_j \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N},$$

there exists a unique sequence $(x_n)_{n \in \mathbb{Z}} \subset X$ satisfying

$$x_{n+1} = A_n x_n + z_n \quad n \in \mathbb{Z},$$

and

$$p_j(x_n - y_n) \leqslant M_j \frac{1 + e^{-w_j}}{1 - e^{-w_j}} \varepsilon_j, \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N}. \tag{3.22}$$

Proof. For $n \in \mathbb{Z}$, we define $f_n \colon X \to X$ by

$$f_n(x) = z_n, \quad x \in X.$$

Observe that (3.4) holds with $c_j = 0$ for $j \in \mathbb{N}$. Hence, (3.3) holds true. The desired conclusion now follows readily from Theorem 1. □

Corollary 2. *Assume that a sequence* $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ *admits an exponential dichotomy with sequences of constants* $(M_j)_{j\in\mathbb{N}}$ *and* $(w_j)_{j\in\mathbb{N}}$*. Then, for every two sequences* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ *and* $(y_n)_{n\in\mathbb{Z}} \subset X$ *such that*

$$p_j(y_{n+1} - A_n y_n) \leqslant \varepsilon_j \quad \text{for } n \in \mathbb{Z} \text{ and } j \in \mathbb{N},$$

there exists a unique sequence $(x_n)_{n\in\mathbb{Z}} \subset X$ *satisfying* (3.1) *and* (3.22)*.*

Proof. The desired conclusion follows directly from Corollary 1 applied to the case when $z_n = 0$ for $n \in \mathbb{Z}$. ☐

3.2. **The continuous time case.** The goal of this section is to establish a version of Theorem 1 for the case of continuous time.

Let $T(t,s)$ be an evolution family and $f_t \colon X \to X$, $t \in \mathbb{R}$ be a family of continuous maps. We assume that there exists a family $U(t,s)\colon X \to X$, $t \geqslant s$ of continuous maps such that

$$U(t,s)x = T(t,s)x + \int_s^t T(t,\tau) f_\tau(U(\tau,s)x)\, d\tau.$$

Moreover, we suppose that $t \mapsto U(t,s)x$ is continuous on $[s,\infty)$ for each $s \in \mathbb{R}$ and $x \in X$. It is easy to verify that

$$U(t,s)U(s,r) = U(t,r), \quad t \geqslant s \geqslant r.$$

Theorem 2. *Let* $T(t,s)$*,* $t \geqslant s$ *be an evolution family that admits an exponential dichotomy. Furthermore, let* $f_t \colon X \to X$*,* $t \in \mathbb{R}$ *be a family of maps with the property that there exists a sequence* $(c_j)_{j\in\mathbb{N}} \subset (0,\infty)$ *such that*

$$p_j(f_t(x) - f_t(y)) \leqslant c_j p_j(x-y), \quad \text{for } t \in \mathbb{R},\ j \in \mathbb{N} \text{ and } x, y \in X. \tag{3.23}$$

Then, provided that c_j*,* $j \in \mathbb{N}$ *are sufficiently small, there exists a sequence* $(\tilde{C}_j)_{j\in\mathbb{N}} \subset (0,\infty)$ *with the property that for each sequence* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ *and a map* $y\colon \mathbb{R} \to X$ *such that*

$$p_j(y(t) - U(t,s)y(s)) \leqslant \varepsilon_j \quad \text{for } j \in \mathbb{N} \text{ and } s \leqslant t \leqslant s+1, \tag{3.24}$$

there exists a map $x\colon \mathbb{R} \to X$ *such that*

$$x(t) = U(t,s)x(s) \quad t \geqslant s, \tag{3.25}$$

and

$$p_j(x(t) - y(t)) \leqslant \tilde{C}_j \varepsilon_j, \quad \text{for } j \in \mathbb{N} \text{ and } t \in \mathbb{R}. \tag{3.26}$$

Proof. Let

$$A_n = T(n+1,n), \quad n \in \mathbb{Z}.$$

It follows readily from (2.6) and (2.7) that

$$p_j(\mathcal{A}(m,n)(\mathrm{Id} - Q(n))x) \leqslant M_j e^{-w_j(m-n)} p_j(x) \quad \text{for } m \geqslant n,\ j \in \mathbb{N},\ x \in X,$$

and

$$p_j(\mathcal{A}(m,n)Q(n)x) \leqslant M_j e^{-w_j(n-m)} p_j(x), \quad \text{for } m \leqslant n,\ j \in \mathbb{N} \text{ and } x \in X.$$

Hence, the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of projections $Q(n)$, $n \in \mathbb{Z}$.

Moreover, for $n \in \mathbb{Z}$ we define $g_n \colon X \to X$ by

$$g_n(x) = \int_n^{n+1} T(n+1,\tau) f_\tau(U(\tau,n)x)\, d\tau, \quad x \in X.$$

Before proceeding, we need the following auxiliary result.

Lemma 1. *For each $j \in \mathbb{N}$ there exists $\tilde{a}_j > 0$ such that*

$$p_j(U(t,s)x - U(t,s)y) \leqslant K_j e^{\tilde{a}_j(t-s)} p_j(x-y), \tag{3.27}$$

for $t \geqslant s$ and $x, y \in X$.

Proof of the lemma. Observe that

$$\begin{aligned} &U(t,s)x - U(t,s)y \\ &= T(t,s)(x-y) + \int_s^t T(t,\tau)(f_\tau(U(\tau,s)x) - f_\tau(U(\tau,s)y))\,d\tau. \end{aligned}$$

Fix an arbitrary $j \in \mathbb{N}$. It follows from (2.5) and (3.23) that

$$\begin{aligned} &p_j(U(t,s)x - U(t,s)y) \\ &\leqslant K_j e^{a_j(t-s)} p_j(x-y) + K_j \int_s^t e^{a_j(t-\tau)} p_j(f_\tau(U(\tau,s)x) - f_\tau(U(\tau,s)y))\,d\tau \\ &\leqslant K_j e^{a_j(t-s)} p_j(x-y) + c_j K_j \int_s^t e^{a_j(t-\tau)} p_j(U(\tau,s)x - U(\tau,s)y)\,d\tau, \end{aligned}$$

for $t \geqslant s$ and $x, y \in X$. From Gronwall's lemma we conclude that (3.27) holds with

$$\tilde{a}_j := a_j + c_j K_j, \quad j \in \mathbb{N}.$$

The proof of the lemma is completed. □

We are now in a position to estimate the Lipschitz norm of g_n for $n \in \mathbb{N}$.

Lemma 2. *We have that*

$$p_j(g_n(x) - g_n(y)) \leqslant \tilde{c}_j p_j(x-y),$$

for $n \in \mathbb{Z}$, $j \in \mathbb{N}$ and $x, y \in X$, where

$$\tilde{c}_j := K_j^2 c_j e^{a_j + \tilde{a}_j}, \quad j \in \mathbb{N}. \tag{3.28}$$

Proof of the lemma. By (2.5), (3.23) and (3.27), we have that

$$\begin{aligned} &p_j(g_n(x) - g_n(y)) \\ &\leqslant K_j \int_n^{n+1} e^{a_j(n+1-\tau)} p_j(f_\tau(U(\tau,n)x) - f_\tau(U(\tau,n)y))\,d\tau \\ &\leqslant K_j c_j \int_n^{n+1} e^{a_j(n+1-\tau)} p_j(U(\tau,n)x - U(\tau,n)y)\,d\tau \\ &\leqslant K_j^2 c_j p_j(x-y) \int_n^{n+1} e^{a_j(n+1-\tau)} e^{\tilde{a}_j(\tau-n)}\,d\tau \\ &\leqslant K_j^2 c_j e^{a_j+\tilde{a}_j} p_j(x-y), \end{aligned}$$

for $n \in \mathbb{Z}$, $j \in \mathbb{N}$ and $x, y \in X$. The proof of the lemma is completed. □

Now, provided that c_j is sufficiently small so that $\tilde{c}_j$ satisfies (3.3) for each $j \in \mathbb{N}$, it follows from Theorem 1 that there exists a sequence $(C_j)_{j\in\mathbb{N}} \subset (0,\infty)$ such that for each two sequences $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ and $(y_n)_{n\in\mathbb{Z}} \subset X$ satisfying

$$p_j(y_{n+1} - A_n y_n - g_n(y_n)) \leqslant \varepsilon_j \quad \text{for } j \in \mathbb{N} \text{ and } n \in \mathbb{Z}, \tag{3.29}$$

there exists a sequence $(x_n)_{n\in\mathbb{Z}} \subset X$ such that

$$x_{n+1} = A_n x_n + g_n(x_n) \quad \text{for } n \in \mathbb{Z}, \tag{3.30}$$

and

$$p_j(x_n - y_n) \leqslant C_j \varepsilon_j, \quad \text{for } j \in \mathbb{N} \text{ and } n \in \mathbb{Z}. \tag{3.31}$$

Let $y\colon \mathbb{R} \to X$ be such that (3.24) holds. Then,

$$p_j(y(n+1) - A_n y(n) - g_n(y(n)) \leqslant \varepsilon_j,$$

for $j \in \mathbb{N}$ and $n \in \mathbb{Z}$. Hence, the sequence $(y(n))_{n\in\mathbb{Z}}$ satisfies (3.29). Thus, there exists a sequence $(x_n)_{n\in\mathbb{Z}} \subset X$ such that (3.30) and (3.31) hold (with $y_n = y(n)$). In particular, (3.30) implies that

$$x_{n+1} = U(n+1, n)x_n, \quad n \in \mathbb{Z}. \tag{3.32}$$

We now define $x\colon \mathbb{R} \to X$ in the following way: take $t \in \mathbb{R}$, choose $n \in \mathbb{Z}$ such that $n \leqslant t < n+1$, and set

$$x(t) := U(t, n)x_n.$$

By (3.32), we have that (3.25) holds. Finally, take an arbitrary $t \in \mathbb{R}$ and choose $n \in \mathbb{Z}$ such that $n \leqslant t < n+1$. Then, for each $j \in \mathbb{N}$ we have that

$$\begin{aligned} p_j(x(t) - y(t)) &= p_j(U(t,n)x_n - y(t)) \\ &\leqslant p_j(U(t,n)x_n - U(t,n)y(n)) + p_j(U(t,n)y(n) - y(t)) \\ &\leqslant K_j e^{\tilde{a}_j} p_j(x_n - y(n)) + \varepsilon_j \\ &\leqslant K_j C_j e^{\tilde{a}_j} \varepsilon_j + \varepsilon_j. \end{aligned}$$

We conclude that (3.26) holds with

$$\tilde{C}_j := K_j C_j e^{\tilde{a}_j} + 1, \quad j \in \mathbb{N}.$$

The proof of the theorem is completed. □

4. Partial linearization

In this section we will be interested in formulating sufficient conditions under which certain nonlinear systems defined on a Fréchet space are topologically conjugated or, more generally, partially conjugated to its linear part.

4.1. The discrete time case.

Given a sequence of $(A_n)_{n\in\mathbb{Z}}$ of operators in $\mathcal{L}(X)$ and a sequence of nonlinear maps $f_n\colon X \to X$, in this subsection we are going to present sufficient conditions under which the nonlinear dynamics given by

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z} \tag{4.1}$$

is topologically conjugated or, more generally, partially conjugated to its linear part

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{4.2}$$

We start by considering the more general case of partial linearization.

Theorem 3. *Assume that the sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ admits an exponential dichotomy with sequences of constants $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$. Let $(c_j)_{j\in\mathbb{N}} \subset [0, \infty)$ be such that*

$$c_j M_j \frac{1 + e^{-w_j}}{1 - e^{-w_j}} < 1, \quad \text{for } j \in \mathbb{N}. \tag{4.3}$$

Finally, let $f_n\colon X \to X$, $n \in \mathbb{Z}$, be a sequence of maps for which there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset [0, \infty)$ such that

$$p_j(f_n(x)) \leqslant \varepsilon_j \tag{4.4}$$

and

$$p_j(f_n(x) - f_n(y)) \leqslant c_j p_j(x-y) \tag{4.5}$$

for every $x, y \in X$, $n \in \mathbb{Z}$ *and* $j \in \mathbb{N}$. *Then, there exists a sequence of one-to-one continuous maps* $H_n \colon R(Q_n) \to X$, $n \in \mathbb{Z}$, *satisfying*

$$H_{n+1}(A_n x) = (A_n + f_n)(H_n(x)) \tag{4.6}$$

for every $x \in R(Q_n)$ *and* $n \in \mathbb{Z}$ *and, moreover,*

$$\sup_{n\in\mathbb{Z}} \sup_{x \in R(Q_n)} p_j(H_n(x) - x) < +\infty \text{ for every } j \in \mathbb{N}. \tag{4.7}$$

Remark 4. In other words what this result is saying is that when restricted to the *unstable* direction of $(A_n)_{n\in\mathbb{N}}$ given by $R(Q_n)$, the system (4.1) is conjugated to its linear part. In particular, the system (4.1) is *partially linearizable*.

Proof. Let $\mathcal{Y}$ denote the space of all sequences $\mathbf{h} = (h_n)_{n\in\mathbb{Z}}$ of continuous maps $h_n \colon R(Q_n) \to X$ such that

$$p_j^\infty(\mathbf{h}) := \sup_{n\in\mathbb{Z}} \sup_{x\in R(Q_n)} p_j(h_n(x)) < +\infty \text{ for every } j \in \mathbb{N}.$$

It is easy to verify that $(\mathcal{Y}, (p_j^\infty)_{j\in\mathbb{N}})$ is a Fréchet space. The sequence of maps $(H_n)_{n\in\mathbb{Z}}$ that we are looking for will have the form

$$H_n = \mathrm{Id}_{R(Q_n)} + h_n$$

with $\mathbf{h} = (h_n)_{n\in\mathbb{Z}} \in \mathcal{Y}$, where $\mathrm{Id}_{R(Q_n)}$ is the identity map on $R(Q_n)$. The construction of these maps will be done inductively in a similar manner to the arguments in the proof of Theorem 1.

Let us consider $\mathbf{h}^1 = (h_n^1)_{n\in\mathbb{Z}}$ given by

$$\begin{aligned} h_n^1(x) &= \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x)) \\ &\quad - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k) Q_k (f_{k-1}(\mathcal{A}(k-1,n)x)), \end{aligned}$$

for every $x \in R(Q_n)$ and $n \in \mathbb{Z}$. Then, by (2.2), (2.3) and (4.4), we have that

$$\begin{aligned} p_j(h_n^1(x)) &\leqslant \sum_{k=-\infty}^{n} p_j(\mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x))) \\ &\quad + \sum_{k=n+1}^{\infty} p_j(\mathcal{A}(n,k) Q_k (f_{k-1}(\mathcal{A}(k-1,n)x))) \\ &\leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)} p_j(f_{k-1}(\mathcal{A}(k-1,n)x)) \\ &\quad + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)} p_j(f_{k-1}(\mathcal{A}(k-1,n)x)) \\ &\leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)} \varepsilon_j + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)} \varepsilon_j \\ &= M_j \frac{1 + e^{-w_j}}{1 - e^{-w_j}} \varepsilon_j, \end{aligned}$$

and thus

$$p_j(h_n^1(x)) \leqslant M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} \varepsilon_j \tag{4.8}$$

for every $x \in R(Q_n)$, $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. Consequently, $\mathbf{h}^1 \in \mathcal{Y}$.

Consider now $\mathbf{h}^2 = (h_n^2)_{n\in\mathbb{Z}}$ given by

$$\begin{aligned} h_n^2(x) = &\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}^1(\mathcal{A}(k-1,n)x))) \\ &- \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}^1(\mathcal{A}(k-1,n)x))), \end{aligned}$$

for every $x \in R(Q_n)$. Proceeding as above it is easy to see that $\mathbf{h}^2 \in \mathcal{Y}$. Moreover, using (2.2), (2.3), (4.5) and (4.8) it follows that

$$p_j(h_n^2(x) - h_n^1(x)) \leqslant c_j \left(M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} \right)^2 \varepsilon_j$$

for every $x \in R(Q_n)$, $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. Consequently,

$$p_j^\infty(\mathbf{h}^2 - \mathbf{h}^1) \leqslant c_j \left(M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} \right)^2 \varepsilon_j \text{ for every } j \in \mathbb{N}.$$

We now proceed by induction: given $l \in \mathbb{N}$, suppose that we have constructed $\mathbf{h}^l = (h_n^l)_{n\in\mathbb{Z}} \in \mathcal{Y}$ such that

$$p_j^\infty(\mathbf{h}^l - \mathbf{h}^{l-1}) \leqslant c_j^{l-1} \left(M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} \right)^l \varepsilon_j, \quad \text{for every } j \in \mathbb{N}. \tag{4.9}$$

We then define the sequence $\mathbf{h}^{l+1} = (h_n^{l+1})_{n\in\mathbb{Z}}$ by

$$\begin{aligned} h_n^{l+1}(x) = &\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}^l(\mathcal{A}(k-1,n)x))) \\ &- \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}^l(\mathcal{A}(k-1,n)x))), \end{aligned}$$

for $x \in R(Q_n)$ and $n \in \mathbb{Z}$. Using (2.2), (2.3) and (4.4) it is again easy to see that $\mathbf{h}^{l+1} \in \mathcal{Y}$. Moreover, (2.2), (2.3), (4.5) and (4.9) imply that

$$p_j(h_n^{l+1}(x) - h_n^l(x)) \leqslant c_j^l \left(M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} \right)^{l+1} \varepsilon_j,$$

for every $x \in R(Q_n)$, $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. In particular,

$$p_j(\mathbf{h}^{l+1} - \mathbf{h}^l) \leqslant c_j^l \left(M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} \right)^{l+1} \varepsilon_j, \quad \text{for every } j \in \mathbb{N}. \tag{4.10}$$

Thus, we have constructed a sequence $(\mathbf{h}^l)_{l\in\mathbb{N}} \in \mathcal{Y}$ which, by (4.3) and (4.10), is a Cauchy sequence in $\mathcal{Y}$ and, in particular, it converges. Let $\mathbf{h} = (h_n)_{n\in\mathbb{Z}} \in \mathcal{Y}$ be such that

$$\mathbf{h} := \lim_{l\to+\infty} \mathbf{h}^l,$$

which in particular gives that

$$h_n(x) = \lim_{l\to\infty} h_n^l(x), \quad \text{for } n \in \mathbb{Z} \text{ and } x \in R(Q_n).$$

Then, given $x \in R(Q_n)$ we have that

$$\begin{aligned} h_n(x) = &\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x))) \\ &- \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x))), \end{aligned}$$

for every $n \in \mathbb{Z}$. Consequently,

$$\begin{aligned} &h_{n+1}(A_n x) \\ &= \sum_{k=-\infty}^{n+1} \mathcal{A}(n+1,k)(\mathrm{Id}-Q_k)(f_{k-1}(\mathcal{A}(k-1,n+1)A_n x + h_{k-1}(\mathcal{A}(k-1,n+1)A_n x))) \\ &\quad - \sum_{k=n+2}^{\infty} \mathcal{A}(n+1,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n+1)A_n x + h_{k-1}(\mathcal{A}(k-1,n+1)A_n x))) \\ &= (\mathrm{Id}-Q_{n+1})(f_n(x+h_n(x))) \\ &\quad + A_n \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x))) \\ &\quad + Q_{n+1}(f_n(x+h_n(x))) \\ &\quad - A_n \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x))) \\ &= f_n(x+h_n(x)) + A_n h_n(x). \end{aligned}$$

Hence,

$$h_{n+1}(A_n x) = A_n h_n(x) + f_n(x+h_n(x))$$

for every $x \in R(Q_n)$ and $n \in \mathbb{Z}$. Then, considering $H_n = \mathrm{Id}_{R(Q_n)} + h_n$ it follows that

$$\begin{aligned} H_{n+1}(A_n x) &= A_n x + h_{n+1}(A_n x) \\ &= A_n x + A_n h_n(x) + f_n(x+h_n(x)) \\ &= (A_n + f_n)(H_n(x)), \end{aligned}$$

for every $x \in R(Q_n)$ and $n \in \mathbb{Z}$. Therefore, H_n satisfies (4.6). Moreover, since $\mathbf{h} = (h_n)_{n\in\mathbb{Z}} \in \mathcal{Y}$, we have that

$$\sup_{n\in\mathbb{Z}} \sup_{x\in R(Q_n)} p_j(H_n(x)-x) = \sup_{n\in\mathbb{Z}} \sup_{x\in R(Q_n)} p_j(h_n(x)) < +\infty$$

for every $j \in \mathbb{N}$ and, consequently, (4.7) is also satisfied. It remains to show that each H_n is an one-to-one map.

Suppose there exist $x_1, x_2 \in R(Q_n)$ such that $H_n(x_1) = H_n(x_2)$ for some $n \in \mathbb{Z}$. Then, using (4.6) we get that

$$H_{n+1}(A_n x_1) = H_{n+1}(A_n x_2).$$

Inductively, we conclude that

$$H_{n+k}(\mathcal{A}(n+k,n)x_1) = H_{n+k}(\mathcal{A}(n+k,n)x_2)$$

for every $k \in \mathbb{N}$. Therefore, recalling the definition of H_{n+k} we get that

$$\mathcal{A}(n+k,n)(x_1-x_2) = h_{n+k}(\mathcal{A}(n+k,n)x_2) - h_{n+k}(\mathcal{A}(n+k,n)x_1)$$

for every $k \geqslant 1$ which implies that

$$p_j(\mathcal{A}(n+k,n)(x_1-x_2)) \leqslant 2p_j^\infty(\mathbf{h}) < +\infty.$$

On the other hand, since $x_1 - x_2 \in R(Q_n)$, it follows by (2.3) that

$$p_j(\mathcal{A}(n+k,n)(x_1-x_2)) \geqslant \frac{1}{M_j} e^{w_j k} p_j(x_1-x_2)$$

for every $k \geqslant 1$ and $j \in \mathbb{N}$. Combining these observations with the fact that $w_j > 0$ we conclude that

$$p_j(x_1-x_2) = 0 \text{ for every } j \in \mathbb{N}.$$

Therefore, recalling (2.1), it follows that $x_1 = x_2$ and H_n is actually one-to-one. This concludes the proof of the theorem. ☐

In the case when the sequence $(A_n)_{n\in\mathbb{Z}}$ is formed by *invertible* operators in $\mathcal{L}(X)$ we can give a stronger version of Theorem 3 which says that systems (4.1) is topologically conjugated to its linear part given by (4.2). In particular, the system (4.1) is *linearizable.*

Theorem 4. *Assume that a sequence $(A_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ is formed by invertible operators in $\mathcal{L}(X)$ and admits an exponential dichotomy with sequences of constants $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$. Let $(c_j)_{j\in\mathbb{N}} \subset [0,\infty)$ be such that*

$$c_j M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} < 1, \quad \textit{for } j \in \mathbb{N}. \tag{4.11}$$

Moreover, let $f_n : X \to X$, $n \in \mathbb{Z}$ be a sequence of maps such that $A_n + f_n$ is an homeomorphism for every $n \in \mathbb{Z}$ and suppose there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset [0,\infty)$ such that

$$p_j(f_n(x)) \leqslant \varepsilon_j \tag{4.12}$$

and

$$p_j(f_n(x) - f_n(y)) \leqslant c_j p_j(x-y) \tag{4.13}$$

for every $x, y \in X$, $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. Then, there exists a sequence of homeomorphisms $H_n : X \to X$, $n \in \mathbb{Z}$, satisfying

$$H_{n+1} \circ A_n = (A_n + f_n) \circ H_n \textit{ for every } n \in \mathbb{Z} \tag{4.14}$$

and

$$\sup_{n\in\mathbb{Z}} \sup_{x\in X} p_j(H_n(x) - x) < +\infty \textit{ for every } j \in \mathbb{N}. \tag{4.15}$$

Remark 5. Suppose that $A \in \mathcal{L}(X)$ is invertible and that Q_j, $j \in \mathbb{N}$ is a sequence of positive numbers such that

$$p_j(A^{-1}x) \leqslant Q_j p_j(x), \quad \text{for } x \in X \text{ and } j \in \mathbb{N}.$$

Furthermore, assume that $f : X \to X$ is a nonlinear map with the property that there exists a sequence c_j, $j \in \mathbb{N}$ of positive numbers such that

$$p_j(f(x) - f(y)) \leqslant c_j p_j(x-y), \quad \text{for } x, y \in X \text{ and } j \in \mathbb{N}.$$

Then, provided that

$$c_j Q_j < 1 \quad \text{for each } j \in \mathbb{N}, \tag{4.16}$$

we have that $A + f$ is a homeomorphism on X. Indeed, fix an arbitrary $y \in X$ and define $F : X \to X$ by

$$F(x) = A^{-1}y - A^{-1}f(x), \quad x \in X.$$

Observe that

$$p_j(F(x_1)-F(x_2)) \leqslant c_j Q_j p_j(x_1-x_2), \quad \text{for } x_1,x_2 \in X \text{ and } j \in \mathbb{N}.$$

Using this together with (4.16), it is easy to show that $F^n(0)$ is a Cauchy sequence in X. Let $x=\lim_{n\to\infty} F^n(0)$. It is easy to verify that $Ax+f(x)=y$. Since y was arbitrary, we have that $A+f$ is surjective.

On the other hand, suppose that there exist $x_1,x_2 \in X$ such that

$$Ax_1+f(x_1)=Ax_2+f(x_2).$$

Then,

$$p_j(x_1-x_2)=p_j(A^{-1}f(x_2)-A^{-1}f(x_1)) \leqslant c_j Q_j p_j(x_1-x_2),$$

for each $j \in \mathbb{N}$. By (4.16), we have that $p_j(x_1-x_2)=0$ for $j\in\mathbb{N}$. Therefore (see (2.1)), $x_1=x_2$, and consequently $A+f$ is injective. We conclude that $A+f$ is a homeomorphism. In particular, this criterion can be used to verify the hypothesis of Theorem 4.

Before we start with the proof, let us fix some notation. For every $n\in\mathbb{Z}$, define $F_n=A_n+f_n$. Then, let us denote by $\mathcal{A}(m,n)$ and $\mathcal{F}(m,n)$ the cocycles associated with $(A_n)_{n\in\mathbb{Z}}$ and $(F_n)_{n\in\mathbb{Z}}$, respectively, which are defined by

$$\mathcal{A}(m,n)=\begin{cases} A_{m-1}\cdots A_n & m>n; \\ \text{Id} & m=n; \\ A_m^{-1}\cdots A_{n-1}^{-1} & m<n; \end{cases}$$

and

$$\mathcal{F}(m,n)=\begin{cases} F_{m-1}\circ\ldots\circ F_n & m>n; \\ \text{Id} & m=n; \\ F_m^{-1}\circ\ldots\circ F_{n-1}^{-1} & m<n. \end{cases}$$

Proof. Let $\tilde{\mathcal{Y}}$ denote the space of all sequences $\mathbf{h}=(h_n)_{n\in\mathbb{Z}}$ of continuous maps $h_n\colon X\to X$ such that

$$p_j^\infty(\mathbf{h}):=\sup_{n\in\mathbb{Z}}\sup_{x\in X} p_j(h_n(x))<+\infty \text{ for every } j\in\mathbb{N}.$$

Then, $(\tilde{\mathcal{Y}},(p_j^\infty)_{j\in\mathbb{N}})$ is a Fréchet space. Now, under the hypothesis of Theorem 4, it is easy to see that the constructions done in the proof of Theorem 3 can be carried out for every $x\in X$ and not just for $x\in R(Q_n)$. This give rise to a sequence of maps $\mathbf{h}=(h_n)_{n\in\mathbb{N}}\in\tilde{\mathcal{Y}}$ such that $H_n=\text{Id}+h_n$ satisfies (4.14) and (4.15) as one can easily observe. It remains to show that each H_n is an homeomorphism. This will be proved by constructing the inverse of H_n explicitly.

Let us consider $\bar{\mathbf{h}}=(\bar{h}_n)_{n\in\mathbb{Z}}$ given by

$$\begin{aligned}\bar{h}_n(x)=&-\sum_{k=-\infty}^{n}\mathcal{A}(n,k)(\text{Id}-Q_k)(f_{k-1}(\mathcal{F}(k-1,n)x))\\ &+\sum_{k=n+1}^{\infty}\mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{F}(k-1,n)x)),\end{aligned}$$

for each $x \in X$ and $n \in \mathbb{Z}$. Proceeding as in the proof of Theorem 3 it is easy to see that $\bar{\mathbf{h}} \in \tilde{\mathcal{Y}}$. Moreover,

$$\begin{aligned}\bar{h}_{n+1}(F_n(x)) &= -\sum_{k=-\infty}^{n+1} \mathcal{A}(n+1,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{F}(k-1,n+1)F_n(x))) \\ &\quad + \sum_{k=n+2}^{\infty} \mathcal{A}(n+1,k)Q_k(f_{k-1}(\mathcal{F}(k-1,n+1)F_n(x))) \\ &= -(\mathrm{Id} - Q_{n+1})(f_n(x)) \\ &\quad - A_n \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{F}(k-1,n)x)) \\ &\quad - Q_{n+1}(f_n(x)) + A_n \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{F}(k-1,n)x)) \\ &= -f_n(x) + A_n\bar{h}_n(x).\end{aligned}$$

Thus,

$$\bar{h}_{n+1}(F_n(x)) = -f_n(x) + A_n\bar{h}_n(x),$$

for every $x \in X$ and $n \in \mathbb{Z}$. Then, considering $\bar{H}_n = \mathrm{Id} + \bar{h}_n$ it is easy to verify that

$$\bar{H}_{n+1} \circ (A_n + f_n) = A_n \circ \bar{H}_n \text{ for every } n \in \mathbb{Z}. \tag{4.17}$$

We now claim that

$$H_n \circ \bar{H}_n = \bar{H}_n \circ H_n = \mathrm{Id} \text{ for every } n \in \mathbb{Z}. \tag{4.18}$$

Indeed, using (4.14) and (4.17) it follows that

$$H_n(\mathcal{A}(n,m)x) = \mathcal{F}(n,m)H_m(x) \tag{4.19}$$

and

$$\bar{H}_n(\mathcal{F}(n,m)x) = \mathcal{A}(n,m)\bar{H}_m(x), \tag{4.20}$$

for every $m, n \in \mathbb{N}$ and $x \in X$. Recalling the definitions of $\bar{H}_n$ and H_n we get that for every $n \in \mathbb{Z}$ and $x \in X$,

$$\begin{aligned}&\bar{H}_n(H_n(x)) \\ &= H_n(x) + \bar{h}_n(H_n(x)) \\ &= x + h_n(x) + \bar{h}_n(H_n(x)) \\ &= x + \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x))) \\ &\quad - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x))) \\ &\quad - \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id} - Q_k)(f_{k-1}(\mathcal{F}(k-1,n)H_n(x))) \\ &\quad + \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{F}(k-1,n)H_n(x))).\end{aligned} \tag{4.21}$$

Now, by (4.19) it follows that

$$\begin{aligned}\mathcal{F}(k-1,n)H_n(x) &= H_{k-1}(\mathcal{A}(k-1,n)x)\\ &= \mathcal{A}(k-1,n)x + h_{k-1}(\mathcal{A}(k-1,n)x),\end{aligned}$$

which combined with (4.21) implies that $\bar{H}_n(H_n(x)) = x$ for every $x \in X$ and $n \in \mathbb{Z}$.

Our objective now is to show that $H_n(\bar{H}_n(x)) = x$ for every $x \in X$ and $n \in \mathbb{Z}$. We start by observing that

$$\begin{aligned}H_n(\bar{H}_n(x)) &= \bar{H}_n(x) + h_n(\bar{H}_n(x))\\ &= x + \bar{h}_n(x) + h_n(\bar{H}_n(x)).\end{aligned}$$

Consequently,

$$H_n(\bar{H}_n(x)) - x = \bar{h}_n(x) + h_n(\bar{H}_n(x)). \tag{4.22}$$

By analyzing the right-hand side of (4.22) we have that

$$\begin{aligned}&\bar{h}_n(x) + h_n(\bar{H}_n(x))\\ &= -\sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(f_{k-1}(\mathcal{F}(k-1,n)x))\\ &\quad + \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{F}(k-1,n)x))\\ &\quad + \sum_{k=-\infty}^{n} \mathcal{A}(n,k)(\mathrm{Id}-Q_k)(f_{k-1}(\mathcal{A}(k-1,n)\bar{H}_n(x) + h_{k-1}(\mathcal{A}(k-1,n)\bar{H}_n(x))))\\ &\quad - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k(f_{k-1}(\mathcal{A}(k-1,n)\bar{H}_n(x) + h_{k-1}(\mathcal{A}(k-1,n)\bar{H}_n(x)))),\end{aligned}$$

for $x \in X$ and $n \in \mathbb{N}$. On the other hand, by using (4.20) we have that

$$\begin{aligned}\mathcal{A}(k-1,n)\bar{H}_n(x) + h_{k-1}(\mathcal{A}(k-1,n)\bar{H}_n(x)) &= H_{k-1}(\mathcal{A}(k-1,n)\bar{H}_n(x))\\ &= H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x)).\end{aligned}$$

Thus, by combining the previous observations and using (2.2), (2.3) and (4.13) we get that

$$\begin{aligned}&p_j(\bar{h}_n(x) + h_n(\bar{H}_n(x)))\\ &\leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)} p_j(f_{k-1}(H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x))) - f_{k-1}(\mathcal{F}(k-1,n)x))\\ &\quad + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)} p_j(f_{k-1}(\mathcal{F}(k-1,n)x) - f_{k-1}(H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x))))\\ &\leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)} c_j p_j(H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x))) - \mathcal{F}(k-1,n)x)\\ &\quad + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)} c_j p_j(H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x))) - \mathcal{F}(k-1,n)x)\end{aligned}$$

for every $j \in \mathbb{N}$. Therefore, using (4.22), it follows that

$$\begin{aligned} & p_j(H_n(\bar{H}_n(x)) - x) \\ & \leqslant \sum_{k=-\infty}^{n} M_j e^{-w_j(n-k)} c_j p_j(H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x))) - \mathcal{F}(k-1,n)x) \\ & \quad + \sum_{k=n+1}^{\infty} M_j e^{-w_j(k-n)} c_j p_j(H_{k-1}(\bar{H}_{k-1}(\mathcal{F}(k-1,n)x))) - \mathcal{F}(k-1,n)x) \end{aligned} \tag{4.23}$$

for every $j \in \mathbb{N}$. Now, since $\mathbf{h} = (h_n)_{n\in\mathbb{N}} \in \tilde{\mathcal{Y}}$ and $\bar{\mathbf{h}} = (\bar{h}_n)_{n\in\mathbb{N}} \in \tilde{\mathcal{Y}}$, it follows from (4.22) that $\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id} := (H_n \circ \bar{H}_n - \mathrm{Id})_{n\in\mathbb{N}} \in \tilde{\mathcal{Y}}$, which combined with (4.23) implies that

$$p_j^\infty(\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id}) \leqslant c_j M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} p_j^\infty(\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id}).$$

Thus, from (4.11) it follows that $p_j^\infty(\mathbf{H} \circ \bar{\mathbf{H}} - \mathrm{Id}) = 0$ for every $j \in \mathbb{N}$ and, consequently (see (2.1)), $H_n(\bar{H}_n(x)) = x$ for every $x \in X$ and $n \in \mathbb{Z}$ proving that (4.18) indeed holds. Thus, since each H_n and $\bar{H}_n$ are continuous, it follows that these are actually homeomorphisms. The proof of Theorem 4 is completed. □

4.2. **The continuous time case.** The goal of this section is to establish the versions of Theorems 3 and 4 in the case of continuous time.

As in Section 3.2, let $T(t,s)$ be an evolution family and $f_t \colon X \to X$, $t \in \mathbb{R}$, be maps for which there exists a family $U(t,s)\colon X \to X$, $t \geqslant s \in \mathbb{R}$, of continuous maps such that

$$U(t,s)x = T(t,s)x + \int_s^t T(t,\tau) f_\tau(U(\tau,s)x)\, d\tau. \tag{4.24}$$

Moreover, we suppose that $t \mapsto U(t,s)x$ is continuous on $[s,\infty)$ for each $s \in \mathbb{R}$ and $x \in X$.

Our first result in this setting is the following.

Theorem 5. *Assume that $T(t,s)$ is an evolution family as above and that it admits an exponential dichotomy with sequences of constants $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$. Let $(c_j)_{j\in\mathbb{N}} \subset [0,\infty)$ be such that*

$$c_j K_j^2 e^{a_j+\tilde{a}_j} M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} < 1, \quad \text{for } j \in \mathbb{N} \tag{4.25}$$

where a_j and $\tilde{a}_j$ come from (2.5) and Lemma 3.27, respectively. Moreover, let $f_t\colon X \to X$, $t \in \mathbb{R}$ be a family of maps as above for which there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset [0,\infty)$ such that

$$p_j(f_t(x)) \leqslant \varepsilon_j \tag{4.26}$$

and

$$p_j(f_t(x) - f_t(y)) \leqslant c_j p_j(x-y) \tag{4.27}$$

for every $x, y \in X$, $t \in \mathbb{R}$ and $j \in \mathbb{N}$. Then, there exists a family of one-to-one continuous maps $H_t\colon R(Q(t)) \to X$, $t \in \mathbb{R}$, satisfying

$$H_t(T(t,s)x) = U(t,s)(H_s(x)) \tag{4.28}$$

for every $t, s \in \mathbb{R}$, $t \geqslant s$ and $x \in R(Q(s))$ and

$$\sup_{t\in\mathbb{R}} \sup_{x\in R(Q(t))} p_j(H_t(x) - x) < +\infty \text{ for every } j \in \mathbb{N}. \tag{4.29}$$

Proof. We will proceed as in the proof of Theorem 2. Let

$$A_n = T(n+1,n), \quad n \in \mathbb{Z}.$$

Then, each A_n is a linear operator in $\mathcal{L}(X)$ and, moreover, it follows readily from (2.6) and (2.7) that the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of projections $Q(n)$, $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ we define $g_n \colon X \to X$ by

$$g_n(x) = \int_n^{n+1} T(n+1,\tau) f_\tau(U(\tau,n)x)\, d\tau, \quad x \in X.$$

It follows from Lemma 2 that each g_n satisfies

$$p_j(g_n(x) - g_n(y)) \leqslant \tilde{c}_j p_j(x-y)$$

for every $x, y \in X$ and $j \in \mathbb{N}$ with constant $\tilde{c}_j := K_j^2 c_j e^{a_j + \tilde{a}_j}$. Moreover, using (2.6), (2.7) and (4.26) it follows that

$$\sup_{x\in X} p_j(g_n(x)) \leqslant \tilde{\varepsilon}_j$$

for every $n \in \mathbb{Z}$ and $j \in \mathbb{N}$ and some sequence of $(\tilde{\varepsilon}_j)_{j\in\mathbb{N}} \subset [0, +\infty)$.

Now, since c_j satisfies (4.25), we have that that $\tilde{c}_j$ satisfies (4.3) for each $j \in \mathbb{N}$. Thus, it follows from Theorem 3 that there exists a sequence of continuous one-to-one maps $H_n \colon Q(n) \to X$ such that

$$H_{n+1}(A_n x) = (A_n + g_n)(H_n(x)) \tag{4.30}$$

for every $x \in R(Q(n))$ and $n \in \mathbb{Z}$. Moreover,

$$\sup_{n\in\mathbb{Z}} \sup_{x\in R(Q(n))} p_j(H_n(x) - x) < +\infty \text{ for every } j \in \mathbb{N}. \tag{4.31}$$

We now define $H_t \colon R(Q(t)) \to X$ in the following way: let $n \in \mathbb{Z}$ be such that $n \leqslant t < n+1$, and set

$$H_t(x) = U(t,n) H_n(T(n,t)x) \tag{4.32}$$

for every $x \in R(Q(t))$. Then, it is easy to see that (4.28) is satisfied. Indeed, given $t, s \in \mathbb{R}$ with $t \geqslant s$, let $m, n \in \mathbb{Z}$ be such that $m \leqslant s < m+1$ and $n \leqslant t < n+1$. Then, using (4.30) we have that $H_n(T(n,m)y) = U(n,m)H_m(y)$ for every $y \in R(Q(m))$. Consequently, given $x \in R(Q(s))$,

$$\begin{aligned}
H_t(T(t,s)x) &= U(t,n)H_n(T(n,t)T(t,s)x) \\
&= U(t,n)H_n(T(n,m)T(m,s)x) \\
&= U(t,n)U(n,m)H_m(T(m,s)x) \\
&= U(t,m)H_m(T(m,s)x) \\
&= U(t,s)U(s,m)H_m(T(m,s)x) \\
&= U(t,s)H_s(x),
\end{aligned}$$

as claimed.

Moreover, given $t \in \mathbb{R}$ let $n \in \mathbb{Z}$ be such that $n \leqslant t < n+1$. Then, for every $x \in R(Q(t))$,

$$U(t,n)T(n,t)x = x + \int_n^t T(t,\tau) f_\tau(U(\tau,n)T(n,t)x)\, d\tau.$$

Consequently, using (2.5) and (4.26),

$$\begin{aligned} p_j(U(t,n)T(n,t)x - x) &\leqslant \int_n^t p_j(T(t,\tau)f_\tau(U(\tau,n)T(n,t)x))\,d\tau \\ &\leqslant K_j e^{a_j}\varepsilon_j, \end{aligned}$$

for every $x \in R(Q(t))$ and $j \in \mathbb{N}$. Combining this fact with Lemma 3.27 we get that for each $j \in \mathbb{N}$ and $x \in R(Q(t))$,

$$\begin{aligned} p_j(H_t(x) - x) &= p_j(U(t,n)H_n(T(n,t)x) - x) \\ &\leqslant p_j(U(t,n)H_n(T(n,t)x) - U(t,n)T(n,t)x)) \\ &\quad + p_j(U(t,n)T(n,t)x - x) \\ &\leqslant K_j e^{\tilde{a}_j} p_j(H_n(T(n,t)x) - T(n,t)x) + K_j e^{a_j}\varepsilon_j. \end{aligned}$$

Therefore,

$$\sup_{x\in R(Q(t))} p_j(H_t(x) - x) \leqslant K_j e^{\tilde{a}_j} \sup_{x\in R(Q(n))} p_j(H_n(x) - x) + K_j e^{a_j}\varepsilon_j.$$

Thus, using (4.31) it follows that

$$\sup_{t\in\mathbb{R}} \sup_{x\in R(Q(t))} p_j(H_t(x) - x) < +\infty$$

for every $j \in \mathbb{N}$ proving (4.29). Finally, if $x_1, x_2 \in R(Q(s))$ are such that $H_s(x_1) = H_s(x_2)$, then using (4.28) we get that

$$H_t(T(t,s)x_1) = H_t(T(t,s)x_2)$$

for every $t \geqslant s$. In particular, this equality holds for $t = m$ with $m \in \mathbb{N}$. Now, by hypothesis we have that $T(m,s)|R(Q(s))$ is one-to-one while by Theorem 3, H_m is also one-to-one. Therefore, $x_1 = x_2$ and H_s is one-to-one for every $s \in \mathbb{R}$. This concludes the proof of the theorem. □

As in the case of discrete time, assuming that the families T and U are invertible, we can get stronger results. More precisely, suppose that for each $t \geqslant s$, $T(t,s)\colon X \to X$ is an isomorphism and for $t < s$ denote $T(t,s) = T(s,t)^{-1}$. Assume moreover that for each $t \geqslant s$ the family $U(t,s)\colon X \to X$ satisfying (4.24) is an homeomorphism and for $t < s$ consider $U(t,s) = U(s,t)^{-1}$. Then we have the following result.

Theorem 6. *Let $T(t,s)$ and $U(t,s)$ be families of maps as above and suppose that that $T(t,s)$ admits an exponential dichotomy with sequences of constants $(M_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$. Let $(c_j)_{j\in\mathbb{N}} \subset [0,\infty)$ be such that*

$$c_j K_j^2 e^{a_j+\tilde{a}_j} M_j \frac{1+e^{-w_j}}{1-e^{-w_j}} < 1, \quad \textit{for } j \in \mathbb{N} \tag{4.33}$$

where a_j and $\tilde{a}_j$ come from (2.5) and Lemma 3.27, respectively. Moreover, let $f_t\colon X \to X$, $t \in \mathbb{R}$ be a family of maps such that the hypothesis about $U(t,s)$ is satisfied and, furthermore, suppose there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset [0,\infty)$ such that

$$p_j(f_t(x)) \leqslant \varepsilon_j \tag{4.34}$$

and

$$p_j(f_t(x) - f_t(y)) \leqslant c_j p_j(x - y) \tag{4.35}$$

for every $x, y \in X$, $t \in \mathbb{R}$ *and* $j \in \mathbb{N}$. *Then, there exists family of homeomorphisms* $H_t \colon X \to X$, $t \in \mathbb{R}$, *satisfying*

$$H_t \circ T(t,s) = U(t,s) \circ H_s \text{ for every } t, s \in \mathbb{R} \tag{4.36}$$

and

$$\sup_{t\in\mathbb{R}} \sup_{x\in X} p_j(H_t(x) - x) < +\infty \text{ for every } j \in \mathbb{N}. \tag{4.37}$$

Proof. As in the proof of Theorem 5, for each $n \in \mathbb{Z}$, let us consider

$$A_n = T(n+1, n), \quad n \in \mathbb{Z}$$

and

$$g_n(x) = \int_n^{n+1} T(n+1, \tau) f_\tau(U(\tau, n)x)\, d\tau, \quad x \in X.$$

Then, each A_n is an invertible operator in $\mathcal{L}(X)$ and, moreover, as observed in the proof of Theorem 5, we have that the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of projections $Q(n)$, $n \in \mathbb{Z}$. Moreover, each g_n satisfies

$$p_j(g_n(x) - g_n(y)) \leqslant \tilde{c}_j p_j(x - y)$$

for every $x, y \in X$ and $j \in \mathbb{N}$ with constant $\tilde{c}_j := K_j^2 c_j e^{a_j + \tilde{a}_j}$ and

$$\sup_{x\in X} p_j(g_n(x)) < \tilde{\varepsilon}_j$$

for every $n \in \mathbb{Z}$ and $j \in \mathbb{N}$ and some sequence of $(\tilde{\varepsilon}_j)_{j\in\mathbb{N}} \subset [0, +\infty)$.

Now, since c_j satisfies (4.33), we have that that $\tilde{c}_j$ satisfy (4.11) for each $j \in \mathbb{N}$. Thus, it follows from Theorem 4 that there exists a sequence of homeomorphisms $H_n \colon X \to X$ such that

$$H_{n+1} \circ A_n = (A_n + g_n) \circ H_n \text{ for every } n \in \mathbb{Z} \tag{4.38}$$

and

$$\sup_{n\in\mathbb{Z}} \sup_{x\in X} p_j(H_n(x) - x) < +\infty \text{ for every } j \in \mathbb{N}. \tag{4.39}$$

Then, defining $H_t \colon X \to X$ by

$$H_t(x) = U(t, n) H_n(T(n, t)x) \tag{4.40}$$

for every $x \in X$ where $n \in \mathbb{Z}$ is such that $n \leqslant t < n + 1$, and proceeding as in the proof of Theorem 5 it follows that (4.36) and (4.37) are satisfied.

Let us now consider $\bar{H}_t \colon X \to X$ given by

$$\bar{H}_t(x) = T(t, n) H_n^{-1}(U(n, t)x)$$

where $n \in \mathbb{Z}$ is such that $n \leqslant t < n+1$. Proceeding again as in the proof of Theorem 5 we can easily see that

$$\bar{H}_t(U(t, s)x) = T(t, s)\bar{H}_s(x)$$

for every $x \in X$. Moreover,

$$\begin{aligned} H_t(\bar{H}_t(x)) &= U(t, n)H_n(T(n, t)\bar{H}_t(x)) \\ &= U(t, n)H_n(T(n, t)T(t, n)H_n^{-1}(U(n, t)x)) \\ &= U(t, n)H_n(H_n^{-1}(U(n, t)x)) \\ &= U(t, n)U(n, t)x) \\ &= x \end{aligned}$$

for every $x \in X$, $t \in [n, n+1)$ and $n \in \mathbb{Z}$. Similarly, $\bar{H}_t(H_t(x)) = x$ for every $x \in X$, $t \in [n, n+1)$ and $n \in \mathbb{Z}$. This proves that each $H_t \colon X \to X$ is an homeomorphism completing the proof of the theorem. □

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