AN ADMISSIBILITY APPROACH TO NONUNIFORM EXPONENTIAL DICHOTOMIES

LUCAS BACKES, DAVOR DRAGIČEVIĆ, AND YONGHUI XIA

Abstract. Recently, Wu and Xia [Proc. Amer. Math. Soc. 151 (2023), 4389-4403] presented a characterization of nonuniform exponential dichotomy via admissibility for difference equations. They have improved previously known results by removing the use of Lyapunov norms and the assumption of bounded growth of the system. However, they have restricted their attention to the case of *finite dimensional* and *invertible* dynamics. In the present work we go one step further and extend their results to the case of possibly *noninvertible* and *infinite dimensional* dynamical systems. We emphasize that our method of proof is different and significantly simpler than the one presented in the aforementioned work.

1. Introduction

Let $(X, |\cdot|)$ be an arbitrary Banach space. Given a nonautonomous dynamical system

$$x_{n+1} = A_n x_n, \quad n \in J, \tag{1}$$

where $A_n \colon X \to X$, $n \in J$, are bounded linear maps and $J \in \{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-\}$ with $\mathbb{Z}^+ = \mathbb{Z} \cap [0, +\infty)$ and $\mathbb{Z}^- = \mathbb{Z} \cap (-\infty, 0]$, for $m, n \in J$, let us consider the evolution operator associated to (1) which is given by

$$\Phi(m,n) = \begin{cases} A_{m-1} \cdots A_n & \text{for } m > n; \\ Id & \text{for } m = n, \end{cases}$$

where Id denotes the identity operator on X. We say that (1) admits a *nonuniform exponential dichotomy* (NED, for short) if the following conditions are satisfied:

(1) there exists a family of projections P_n, $n \in J$, such that

$$A_n P_n = P_{n+1} A_n; \tag{2}$$

(2) $A_n|_{KerP_n} \colon KerP_n \to KerP_{n+1}$ is an invertible operator for each $n \in J$;

(3) there exist $D > 0$, $0 < \alpha < 1$ and $\varepsilon \geq 1$ such that

$$|\Phi(m,n)P_n| \leq D\alpha^{m-n}\varepsilon^{|n|} \quad \text{for } m \geq n \tag{3}$$

and

$$|\Phi(m,n)(Id - P_n)| \leq D\alpha^{n-m}\varepsilon^{|n|} \quad \text{for } m \leq n \tag{4}$$

where

$$\Phi(m,n) := \left(\Phi(n,m)|_{KerP_m}\right)^{-1} \colon KerP_n \to KerP_m,$$

for $m \leq n$.

2020 *Mathematics Subject Classification.* Primary: 34D09, 39A06; Secondary: 37D25.

Key words and phrases. Admissibility; Nonautonomous difference equations; Nonuniform exponential dichotomy.

L. Backes was partially supported by a CNPq-Brazil PQ fellowship under Grant No. 307633/2021-7. D. Dragičević was supported in part by University of Rijeka under the project uniri-iskusni-prirod-23-98 3046. Y. Xia was supported by the National Natural Science Foundation of China (No. 11931016) and Natural Science Foundation of Zhejiang Province (No. LZ24A010006).

We note that in the particular case when $\varepsilon = 1$, we recover the classical concept of (uniform) exponential dichotomy.

The notion of NED along with its counterpart in smooth dynamics known as *nonuniform hyperbolicity* is ubiquitous in dynamical systems (see for instance [6, 7]). On the other hand, in general, it may be very difficult to verify directly if a system admits a NED. Therefore, an important problem is to find different characterizations of this property.

In this paper we will turn our attention to the problem of characterizing NED in terms of the admissibility of certain pairs of Banach spaces. We say that the pair (Y, Z) is *(properly) admissible* for Eq. (1), where Y and Z are subspaces of X^J, if for every sequence $(y_n)_{n\in J}$ in Y there exists a (unique) sequence $(x_n)_{n\in J}$ in Z such that

$$x_{n+1} = A_n x_n + y_{n+1}, \qquad \text{for all } n \in J.$$

We emphasize that the characterizations of *uniform* asymptotic behaviours (uniform exponential stability or dichotomy) of continuous and discrete dynamical systems in terms of admissibility have a long history that goes back to the pioneering works of Perron [23] and Li [17]. These were followed by seminal contributions of Massera and Schäffer [18, 19], Coffman and Schäffer [8], Dalec'kiĭ and Kreĭn [10], Coppel [9] and Henry [13]. For more recent contributions, we refer to [14, 15, 16, 21, 22, 24, 26] and references therein. Finally, for a comprehensive overview of this line of research we refer to the book [5].

The first characterizations of NED's via admissibility relied on the use of the so-called Lyapunov norms which transform nonuniform behaviour into a uniform one (see [2, 3, 4, 31]). For earlier work which was concerned with some different concepts of nonuniform dichotomies we refer to [20, 25, 28]. On the other hand, we stress that Lyapunov norms are difficult to construct without knowing that our dynamics exhibits nonuniform exponential behaviour, and consequently it was natural to explore the relationship between nonuniform behaviour and admissibility avoiding the use of such norms. In this direction, Zhou and Zhang [30] obtained a complete description of NED's using admissibility of two pairs of weighted sequence spaces. We stress that the results in [30] deal with the case when coefficients A_n in (1) are invertible linear operators on $X = \mathbb{R}^d$. Moreover, it was assumed that (1) exhibits the so-called bounded growth property (see [30, (2.3)]). In the recent work [27], Wu and Xia obtained characterizations of NED's in terms of admissibility analogous to those in [30] but eliminating the assumption of bounded growth.

However, the results in [27] were still restricted to the case of finite dimensional and invertible dynamics. The main objective of the present work is to go one step further and extend their results to the case of possibly *noninvertible* and *infinite dimensional* dynamics. Moreover, we emphasize that our method of proofs is different and significantly simpler than the one presented in [27]. The importance of our results stems from the fact that NEDs are ubiquitous in the context of ergodic theory even in the infinite-dimensional case when the appropriate versions of the multiplicative ergodic theorem can be applied (see for example [1, Proposition 3.2]). In this setting (as well as in many others) the assumption that the dynamics is invertible is way too restrictive, and returns us to the finite-dimensional case. We stress that similar results were obtained in [11, 12] where it was shown that NED's can be characterized in terms of admissibility of *three* pairs of spaces. On the other hand, in the present work we show that the same can be achieved by using only two pairs.

2. Main results

For $\delta > 0$, let us consider

$$\mathcal{L}_\delta^\infty(J) := \left\{ f\colon J \to X : \sup_{k \in J} \left(|f(k)| \delta^{-|k|} \right) < +\infty \right\},$$

which is a Banach space with respect to the norm

$$|f|_{\delta,J} := \sup_{k \in J} \left(|f(k)| \delta^{-|k|} \right).$$

Moreover, for $m \in \mathbb{Z}^+$ let

$$U_{\delta,\mathbb{Z}^+}(m) := \left\{ v \in X : \sup_{k \geq m} (|\Phi(k,m)v| \delta^{-k}) < +\infty \right\}.$$

Clearly, $U_{\delta,\mathbb{Z}^+}(m)$ is a subspace of X.

The following is a generalization of [27, Lemma 2.1] to the case of *noninvertible* dynamics on *arbitrary* Banach spaces. We note that our arguments are much simpler than those in [27].

Proposition 2.1. *Assume that there are $0 < \delta \leq \gamma$ such that the pair $(\mathcal{L}_\delta^\infty(\mathbb{Z}^+), \mathcal{L}_\gamma^\infty(\mathbb{Z}^+))$ is admissible. Furthermore, suppose that $U_{\gamma,\mathbb{Z}^+}(0)$ is closed and complemented in X. Then, there is a sequence of projections P_n, $n \in \mathbb{Z}^+$ on X such that (2) holds for $n \in \mathbb{Z}^+$ with $A_n|_{KerP_n}\colon KerP_n \to KerP_{n+1}$ being invertible, and a constant $D > 0$ such that*

$$|\Phi(n,m)P_m| \leq D\gamma^{n-m} \left(\frac{\gamma}{\delta} \right)^m \quad n \geq m, \tag{5}$$

and

$$|\Phi(n,m)(Id - P_m)| \leq D\gamma^{n-m} \left(\frac{\gamma}{\delta} \right)^m \quad n < m. \tag{6}$$

Proof. We divide the proof of Proposition 2.1 into several lemmas. To this end, let $Z \subset X$ be a closed subspace of X such that

$$X = U_{\gamma,\mathbb{Z}^+}(0) \oplus Z. \tag{7}$$

Lemma 2.2. *For each $f \in \mathcal{L}_\delta^\infty(\mathbb{Z}^+)$ there exists a unique $x_f \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^+)$ such that $x_f(0) \in Z$ and*

$$x_f(n+1) = A_n x_f(n) + f(n), \quad n \in \mathbb{Z}^+. \tag{8}$$

Proof of the Lemma 2.2. Indeed, we know that there exists $x \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^+)$ such that

$$x(n+1) = A_n x(n) + f(n), \quad n \in \mathbb{Z}^+. \tag{9}$$

Write $x(0) = v_1 + v_2$ where $v_1 \in U_{\gamma,\mathbb{Z}^+}(0)$ and $v_2 \in Z$. Define

$$x_f(n) := x(n) - \Phi(n,0)v_1, \quad n \in \mathbb{Z}^+.$$

Then, $x_f \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^+)$, $x_f(0) = v_2 \in Z$ and (8) follows from (9). Therefore, we have established the existence of x_f.

We now prove that x_f is unique. To this end, suppose that $\tilde{x}_f\colon \mathbb{Z}^+ \to X$ belongs to $\mathcal{L}_\gamma^\infty(\mathbb{Z}^+)$, $\tilde{x}_f(0) \in Z$ and satisfies

$$\tilde{x}_f(n+1) = A_n \tilde{x}_f(n) + f(n), \quad n \in \mathbb{Z}^+. \tag{10}$$

By (8) and (10), we have that

$$x_f(n) - \tilde{x}_f(n) = \Phi(n,0)(x_f(0) - \tilde{x}_f(0)), \quad n \in \mathbb{Z}^+.$$

Since $x_f - \tilde{x}_f \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^+)$, we conclude that $x_f(0) - \tilde{x}_f(0) \in U_{\gamma,\mathbb{Z}^+}(0)$. On the other hand, $x_f(0) - \tilde{x}_f(0) \in Z$. Then, (7) yields that $x_f(0) = \tilde{x}_f(0)$ and therefore $x_f \equiv \tilde{x}_f$. ∎

Lemma 2.3. *We have that*

$$X = U_{\gamma,\mathbb{Z}^+}(m) \oplus \Phi(m,0)Z, \quad m \in \mathbb{Z}^+. \tag{11}$$

Proof of the Lemma 2.3. For $m = 0$ there is nothing to prove as the desired conclusion follows from (7). Take $m > 0$ and $v \in X$. We define $f\colon \mathbb{Z}^+ \to X$ by

$$f(n) = \begin{cases} 0 & n \neq m-1 \\ v & n = m-1. \end{cases}$$

Then, $f \in \mathcal{L}^\infty_\delta(\mathbb{Z}^+)$. By Lemma 2.2, there is a unique $x_f \in \mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$ such that $x_f(0) \in Z$ and that (8) holds. Then,

$$x_f(m) - A_{m-1}x_f(m-1) = v$$

and

$$x_f(n) = A_{n-1}x_f(n-1) \quad n \neq m.$$

In particular,

$$A_{m-1}x_f(m-1) = \Phi(m,0)x_f(0) \in \Phi(m,0)Z.$$

Moreover,

$$x_f(n) = \Phi(n,m)x_f(m) \quad n \geq m,$$

and thus, since $x_f \in \mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$, we have that $x_f(m) \in U_{\gamma,\mathbb{Z}^+}(m)$. Hence, $v \in U_{\gamma,\mathbb{Z}^+}(m) + \Phi(m,0)Z$.

Suppose now that $v \in U_{\gamma,\mathbb{Z}^+}(m) \cap \Phi(m,0)Z$. Thus, there exists $z \in Z$ such that $v = \Phi(m,0)z$. Set

$$x(n) := \Phi(n,0)z, \quad n \in \mathbb{Z}^+.$$

Then, $x \in \mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$, $x(0) \in Z$ and (9) holds with $f \equiv 0$. From Lemma 2.2 we conclude that $x \equiv 0$ which implies that $v = 0$. This completes the proof. □

Set

$$Z(m) := \Phi(m,0)Z, \quad m \in \mathbb{Z}^+.$$

Clearly,

$$A_m U_{\gamma,\mathbb{Z}^+}(m) \subset U_{\gamma,\mathbb{Z}^+}(m+1) \quad \text{and} \quad A_m Z(m) = Z(m+1), \tag{12}$$

for $m \in \mathbb{Z}^+$.

Lemma 2.4. *For $m \in \mathbb{Z}^+$, $A_m|_{Z(m)}\colon Z(m) \to Z(m+1)$ is invertible.*

Proof of the lemma 2.4. The surjectivity is obvious. Suppose that there exists $v \in Z(m)$ such that $A_m v = 0$. We write $v = \Phi(m,0)z$ for $z \in Z$. Set

$$x(n) := \Phi(n,0)z, \quad n \in \mathbb{Z}^+.$$

Since $x(n) = 0$ for $n > m$, we have that $x \in \mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$. Moreover, $x(0) \in Z$ and (9) holds with $f \equiv 0$. Thus, $x \equiv 0$ and $v = 0$ and $A_m|_{Z(m)}\colon Z(m) \to Z(m+1)$ is injective. The proof is complete. □

By Lemma 2.2, we may define a map from $\mathcal{L}^\infty_\delta(\mathbb{Z}^+)$ into $\mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$ by $f \mapsto x_f$. It is easy to see that this is a linear operator. Moreover, the following holds.

Lemma 2.5. *The linear operator $f \mapsto x_f$ is bounded.*

Proof of the Lemma 2.5. We will prove that $f \mapsto x_f$ is a closed operator, which due to the closed graph theorem implies the desired result. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^\infty_\delta(\mathbb{Z}^+)$ that converges to f in $\mathcal{L}^\infty_\delta(\mathbb{Z}^+)$. Moreover, assume that the sequence $(x_{f_n})_{n\in\mathbb{N}}$ converges to x in $\mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$. We have that

$$x_{f_n}(m+1) = A_m x_{f_n}(m) + f_n(m), \quad m \in \mathbb{Z}^+. \tag{13}$$

Since $f_n \to f$ in $\mathcal{L}^\infty_\delta(\mathbb{Z}^+)$ and $x_{f_n} \to x$ in $\mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$, we conclude that $f_n(m) \to f(m)$ and $x_{f_n}(m) \to x(m)$ for each $m \in \mathbb{Z}^+$. Since $x_{f_n}(0) \in Z$ for each $n \in \mathbb{N}$, we have that $x(0) \in Z$ (since Z is closed). Moreover, by passing to the limit when $n \to \infty$ in (13) we have that

$$x(m+1) = A_m x(m) + f(m), \quad m \in \mathbb{Z}^+.$$

Therefore, $x_f \equiv x$ and the desired conclusion follows. □

For $m \in \mathbb{Z}^+$, let $P_m \colon X \to U_{\gamma,\mathbb{Z}^+}(m)$ be the projections associated with the splitting (11). Clearly, (12) implies (2). Using the notation as in the proof of Lemma 2.3 we have for $m > 0$ that $P_m v = x_f(m)$. Thus,

$$|P_m v| = |x_f(m)| \le \gamma^m |x_f|_{\gamma,\mathbb{Z}^+} \le K\gamma^m |f|_{\delta,\mathbb{Z}^+} = K\delta(\gamma/\delta)^m |v|, \tag{14}$$

where $K > 0$ denotes the norm of the operator $f \mapsto x_f$. Take now $v \in U_{\gamma,\mathbb{Z}^+}(m)$, $m > 0$ and define $f \colon \mathbb{Z}^+ \to X$ by

$$f(n) = \begin{cases} 0 & n \neq m-1 \\ v & n = m-1. \end{cases} \tag{15}$$

Then, $f \in \mathcal{L}^\infty_\gamma(\mathbb{Z}^+)$. It is easy to see that

$$x_f(n) = \begin{cases} \Phi(n,m)v & n \ge m; \\ 0 & n < m. \end{cases}$$

Hence,

$$|\Phi(n,m)v| \le \gamma^n |x_f|_{\gamma,\mathbb{Z}^+} \le K\gamma^n |f|_{\delta,\mathbb{Z}^+} \le K\delta\gamma^n\delta^{-m}|v| = K\delta\gamma^{n-m}\left(\frac{\gamma}{\delta}\right)^m |v|,$$

for $n \ge m$. We conclude (see (14)) that

$$|\Phi(n,m)P_m v| \le K^2\delta^2\gamma^{n-m}\left(\frac{\gamma}{\delta}\right)^{2m}|v|, \quad n \ge m > 0.$$

We now consider the case $m = 0$. If $n > 0$, then

$$|\Phi(n,0)P_0 v| = |\Phi(n,1)P_1 A_0 v| \le K^2\delta^2\gamma^{n-1}\frac{\gamma}{\delta}|A_0 v| \le K^2\delta|A_0|\gamma^n|v|.$$

On the other hand,

$$|\Phi(0,0)P_0 v| \le |P_0| \cdot |v|.$$

Hence,

$$|\Phi(n,m)P_m v| \le D_1\gamma^{n-m}\left(\frac{\gamma}{\delta}\right)^{2m}|v| \quad n \ge m, \tag{16}$$

where

$$D_1 := \max\{|P_0|, K^2\delta^2, K^2\delta|A_0|\}.$$

Take now $v \in Z(m)$ with $m > 0$. We define $f \colon \mathbb{Z}^+ \to X$ by

$$f(n) = \begin{cases} -v & n = m-1 \\ 0 & n \neq m-1. \end{cases}$$

Then, $f \in \mathcal{L}^\infty_\delta(\mathbb{Z}^+)$ and the corresponding x_f is given by

$$x_f(n) = \begin{cases} \Phi(n,m)v & n \le m-1 \\ 0 & n \ge m. \end{cases}$$

Hence,

$$|\Phi(n,m)v| \le \gamma^n |x_f|_{\gamma,\mathbb{Z}^+} \le K\gamma^n |f|_{\delta,\mathbb{Z}^+} \le K\delta\gamma^n\delta^{-m}|v| = K\delta\gamma^{n-m}\left(\frac{\gamma}{\delta}\right)^m |v|$$

for $n < m$. Thus, by (14) we have

$$|\Phi(n,m)(Id - P_m)v| \le K(1+K\delta)\delta\gamma^{n-m}\left(\frac{\gamma}{\delta}\right)^{2m}|v|, \tag{17}$$

for $n < m$.

Finally, for $v \in X$ and $m > 0$, let f be given by (15). Moreover, let $x\colon \mathbb{Z}^+ \to X$ be given by

$$x(n) = \begin{cases} \Phi(n,m)P_m v & n \ge m; \\ -\Phi(n,m)(Id - P_m)v & n < m. \end{cases}$$

It is easy to verify that (9) holds. Moreover, (16) and (17) imply that $x \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^+)$. Since $x(0) \in Z$ we conclude that $x = x_f$. Then,

$$|\Phi(n,m)P_m v| \le \gamma^n |x_f|_{\gamma,\mathbb{Z}^+} \le K\gamma^n |f|_{\delta,\mathbb{Z}^+} = K\delta\gamma^n\delta^{-m}|v| = K\delta\gamma^{n-m}\left(\frac{\gamma}{\delta}\right)^m |v|,$$

for $n \ge m$, which together with (16) applied for $m = 0$ yields (5). Similarly, one can prove (6). This ends the proof of Proposition 2.1. □

Remark 2.6. *We note that in comparison to* [27, Lemma 2.1] *in* (5) *and* (6) *we have* $(\gamma/\delta)^m$ *instead of* $(\gamma/\delta)^{2m}$.

We now state our first main result.

Theorem 2.7. *Assume that there are* $0 < \delta_i \le \gamma_i$, $i \in \{1,2\}$ *with* $\gamma_1 < 1$ *and* $\gamma_2 > 1$ *such that the pairs* $(\mathcal{L}_{\delta_i}^\infty(\mathbb{Z}^+), \mathcal{L}_{\gamma_i}^\infty(\mathbb{Z}^+))$ *are admissible for* $i = 1,2$. *Moreover, suppose that*

$$U_{\gamma_1,\mathbb{Z}^+}(m) = U_{\gamma_2,\mathbb{Z}^+}(m) \quad \text{for } m \in \mathbb{Z}^+, \tag{18}$$

and that $U_{\gamma_1,\mathbb{Z}^+}(0)$ *is closed and complemented. Then,* $(A_n)_{n\in\mathbb{Z}^+}$ *admits NED.*

Proof. Let Z be a closed subspace of X such that

$$X = U_{\gamma_1,\mathbb{Z}^+}(0) \oplus Z = U_{\gamma_2,\mathbb{Z}^+}(0) \oplus Z.$$

Then, it follows from the proof of the previous proposition that we can choose the same projections P_m, $m \in \mathbb{Z}^+$ associated to both admissible pairs. In particular, we have

$$|\Phi(n,m)P_m| \le D\gamma_1^{n-m}\left(\frac{\gamma_1}{\delta_1}\right)^m \quad n \ge m,$$

and

$$|\Phi(n,m)(Id - P_m)| \le D'\gamma_2^{n-m}\left(\frac{\gamma_2}{\delta_2}\right)^m \quad n < m.$$

This readily implies the desired conclusion. □

Remark 2.8. *In the case when operators* A_m *are invertible, one can easily verify that* $U_{\gamma,\mathbb{Z}^+}(m) = \Phi(m,0)(U_{\gamma,\mathbb{Z}^+}(0))$ *for* $m \in \mathbb{Z}^+$. *Thus, in this case* (18) *can be replaced by the requirement that* $U_{\gamma_1,\mathbb{Z}^+}(0) = U_{\gamma_2,\mathbb{Z}^+}(0)$. *Moreover, when* X *is finite-dimensional, the assumption that* $U_{\gamma_1,\mathbb{Z}^+}(0)$ *is closed and complemented is automatically satisfied. Finally, when* X *is a Hilbert space, it is sufficient to assume that* $U_{\gamma_1,\mathbb{Z}^+}(0)$ *is closed.*

Let us now discuss the case of dichotomies on $\mathbb{Z}^-$. For $m \in \mathbb{Z}^-$, let $\tilde{U}_{\delta,\mathbb{Z}^-}(m)$ consist of all $v \in X$ with the property that there is a sequence $(x(n))_{n\le m} \subset X$ such that $x(m) = v$, $x(n) = A_{n-1}x(n-1)$ for $n \le m$ and $\sup_{n\le m}(|x(n)|\delta^{-|n|}) < +\infty$. Then, $\tilde{U}_{\delta,\mathbb{Z}^-}(m)$ is a subspace of X.

Proposition 2.9. *Assume that there are* $0 < \delta \le \gamma$ *such that:*

(a) *the pair* $(\mathcal{L}_\delta^\infty(\mathbb{Z}^-), \mathcal{L}_\gamma^\infty(\mathbb{Z}^-))$ *is admissible;*
(b) $\tilde{U}_{\gamma,\mathbb{Z}^-}(0)$ *is closed and complemented in* X;
(c) *if* $x \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^-)$, $x(n) = A_{n-1}x(n-1)$ *for* $n \le 0$ *and* $x(0) = 0$, *then* $x \equiv 0$.

Then, there is a sequence of projections P_m^-, $m \in \mathbb{Z}^-$ on X such that

$$A_m P_m^- = P_{m+1}^- A_m \quad m \le -1 \tag{19}$$

with $A_m|_{KerP_m} \colon KerP_m \to KerP_{m+1}$ being invertible, and a constant $D > 0$ such that

$$|\Phi(n,m)P_m^-| \le D\gamma^{m-n}\left(\frac{\gamma}{\delta}\right)^{|m|} \quad n \ge m, \tag{20}$$

and

$$|\Phi(n,m)(Id - P_m^-)| \le D\gamma^{m-n}\left(\frac{\gamma}{\delta}\right)^{|m|} \quad n < m. \tag{21}$$

Proof. Let Z be a closed subspace of X such that

$$X = Z \oplus \tilde{U}_{\gamma,\mathbb{Z}^-}(0). \tag{22}$$

Lemma 2.10. *For each $f \in \mathcal{L}_\delta^\infty(\mathbb{Z}^-)$ there exists a unique $x_f \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^-)$ such that $x_f(0) \in Z$ and*

$$x_f(n+1) = A_n x_f(n) + f(n), \quad n \le -1. \tag{23}$$

Proof of the Lemma 2.10. The existence of x_f can be established by arguing as in the proof of Lemma 2.2. The uniqueness follows from (c). □

For $m \in \mathbb{Z}^-$, let

$$Z(m) := \{v \in X : \ \Phi(0,m)v \in Z\}, \quad m \in \mathbb{Z}^-.$$

Note that $Z(m)$ is a subspace of X and $Z(0) = Z$.

Lemma 2.11. *For $m \in \mathbb{Z}^-$,*

$$X = Z(m) \oplus \tilde{U}_{\gamma,\mathbb{Z}^-}(m). \tag{24}$$

Proof of the Lemma 2.11. For $m = 0$ there is nothing to prove as the desired conclusion follows from (22). Take now $m < 0$. For $v \in X$, we define $f \colon \mathbb{Z}^- \to X$ by

$$f(n) = \begin{cases} v & n = m-1; \\ 0 & n \ne m-1. \end{cases}$$

Then, $f \in \mathcal{L}_\delta^\infty(\mathbb{Z}^-)$. Let $x_f \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^-)$ be given by Lemma 2.10. Then,

$$x_f(m) - A_{m-1}x_f(m-1) = v$$

and

$$x_f(n) = A_{n-1}x_f(n-1), \quad n \ne m.$$

In particular, $x_f(0) = \Phi(0,m)x_f(m)$ yielding that $x_f(m) \in Z(m)$ (since $x_f(0) \in Z$). Moreover, $A_{m-1}x_f(m-1) \in \tilde{U}_{\gamma,\mathbb{Z}^-}(m)$. Hence, $v \in Z(m) + \tilde{U}_{\gamma,\mathbb{Z}^-}(m)$.

Take now $v \in Z(m) \cap \tilde{U}_{\gamma,\mathbb{Z}^-}(m)$. Then, there exists a sequence $(x(n))_{n \le m} \subset X$ such that $x(m) = v$, $x(n) = A_{n-1}x(n-1)$ for $n \le m$ and $\sup_{n\le m}(|x(n)|\gamma^{-|n|}) < +\infty$. We define

$$\tilde{x}(n) := \begin{cases} x(n) & n \le m; \\ \Phi(n,m)v & n > m. \end{cases}$$

Then, $\tilde{x} \in \mathcal{L}_\gamma^\infty(\mathbb{Z}^-)$ and

$$\tilde{x}(n) = A_{n-1}\tilde{x}(n-1), \quad n \le 0.$$

Since $\tilde{x}(0) \in Z$, by Lemma 2.10 we have that $\tilde{x} \equiv 0$, and thus $v = 0$. □

Lemma 2.12. *For $m \le -1$, $A_m|_{\tilde{U}_{\gamma,\mathbb{Z}^-}(m)} \colon \tilde{U}_{\gamma,\mathbb{Z}^-}(m) \to \tilde{U}_{\gamma,\mathbb{Z}^-}(m+1)$ is invertible.*

Proof of the Lemma 2.12. The surjectivity is obvious. Let $v \in \tilde{U}_{\gamma,\mathbb{Z}^-}(m)$ be such that $A_m v = 0$. Let $(x(n))_{n\le m} \subset X$ be a sequence such that $x(m) = v$, $x(n) = A_{n-1}x(n-1)$ for $n \le m$ and $\sup_{n\le m}(|x(n)|\gamma^{-|n|}) < +\infty$. We define

$$\tilde{x}(n) := \begin{cases} x(n) & n \le m; \\ 0 & n > m. \end{cases}$$

Then, $\tilde{x} \in \mathcal{L}^\infty_\gamma(\mathbb{Z}^-)$, $\tilde{x}(0) \in Z$ and $\tilde{x}(n) = A_{n-1}\tilde{x}(n-1)$ for $n \le 0$. By Lemma 2.10, we have that $\tilde{x} \equiv 0$. Hence, $v = 0$ and $A_m|_{\tilde{U}_{\gamma,\mathbb{Z}^-}(m)}$ is injective which together with the previous observation yields our claim. □

Using Lemma 2.10 we may define a map from $\mathcal{L}^\infty_\delta(\mathbb{Z}^-)$ into $\mathcal{L}^\infty_\gamma(\mathbb{Z}^-)$ by $f \mapsto x_f$. It is easy to see that this is a linear operator and, moreover, by proceeding as in the proof of Lemma 2.5 we get the following result.

Lemma 2.13. *The linear operator $f \mapsto x_f$ is bounded.*

Let $P^-_m \colon X \to Z(m)$, $m \le 0$ be the projections associated to the splitting (24). Then, using the notation as in the proof of Lemma 2.11 we have that $P^-_m v = x_f(m)$ and, consequently,

$$|P^-_m v| = |x_f(m)| \le \gamma^{|m|}|x_f|_{\gamma,\mathbb{Z}^-} \le K\gamma^{|m|}|f|_{\delta,\mathbb{Z}^-} = \frac{K}{\delta}\left(\frac{\gamma}{\delta}\right)^{|m|}|v|, \tag{25}$$

for $m \le 0$, where $K > 0$ is the norm of the operator $f \mapsto x_f$. We now take $v \in Z(m)$ and define $f \colon \mathbb{Z}^- \to X$ by

$$f(n) = \begin{cases} v & n = m-1; \\ 0 & n \ne m-1. \end{cases}$$

Then, $f \in \mathcal{L}^\infty_\delta(\mathbb{Z}^-)$ and x_f is given by

$$x_f(n) = \begin{cases} \Phi(n,m)v & n \ge m; \\ 0 & n < m. \end{cases}$$

Therefore,

$$|\Phi(n,m)v| \le \gamma^{|n|}|x_f|_{\gamma,\mathbb{Z}^-} \le K\gamma^{-n}|f|_{\delta,\mathbb{Z}^-} = \frac{K}{\delta}\gamma^{-n}\delta^m|v|,$$

and thus

$$|\Phi(n,m)v| \le \frac{K}{\delta}\gamma^{m-n}\left(\frac{\gamma}{\delta}\right)^{|m|} \quad n \ge m.$$

Consequently (see (25)),

$$|\Phi(n,m)P^-_m| \le C\gamma^{m-n}\left(\frac{\gamma}{\delta}\right)^{2|m|} \quad n \ge m, \tag{26}$$

where $C := K^2/\delta^2 > 0$.

Take now $v \in \tilde{U}_{\gamma,\mathbb{Z}^-}(m)$. Let $f \colon \mathbb{Z}^- \to X$ be given by

$$f(n) = \begin{cases} -v & n = m-1; \\ 0 & n \ne m-1. \end{cases}$$

Then, $f \in \mathcal{L}^\infty_\delta(\mathbb{Z}^-)$ and the corresponding x_f is given by

$$x_f(n) = \begin{cases} \Phi(n,m)v & n < m; \\ 0 & n \ge m. \end{cases}$$

Consequently,

$$|\Phi(n,m)v| \le \gamma^{-n}|x_f|_{\gamma,\mathbb{Z}^-} \le K\gamma^{-n}|f|_{\delta,\mathbb{Z}^-} = \frac{K}{\delta}\gamma^{-n}\delta^m|v|,$$

and thus

$$|\Phi(n,m)(Id-P^-_m)|\leq C'\gamma^{m-n}\left(\frac{\gamma}{\delta}\right)^{2|m|} \quad n<m, \tag{27}$$

for some $C'>0$. Finally, proceeding as we did in the end of the proof of Proposition 2.1, we can improve the exponent "$2|m|$" to "$|m|$" in (26) and (27). Hence, (20) and (21) hold and the proof of the proposition is complete. □

The proof of the following result is analogous to the proof of Theorem 2.7.

Theorem 2.14. *Assume that there are $0<\delta_i\leq\gamma_i$, $i\in\{1,2\}$ with $\gamma_1<1$ and $\gamma_2>1$ such that:*

(a) *the pairs $(\mathcal{L}^\infty_{\delta_i}(\mathbb{Z}^-),\mathcal{L}^\infty_{\gamma_i}(\mathbb{Z}^-))$ are admissible for $i=1,2$;*

(b) *for $m\in\mathbb{Z}^-$,*

$$\tilde{U}_{\gamma_1,\mathbb{Z}^-}(m)=\tilde{U}_{\gamma_2,\mathbb{Z}^-}(m); \tag{28}$$

(c) *$\tilde{U}_{\gamma_1,\mathbb{Z}^-}(0)$ is closed and complemented;*

(d) *if $x\in\mathcal{L}^\infty_{\gamma_1}(\mathbb{Z}^-)$, $x(n)=A_{n-1}x(n-1)$ for $n\leq 0$ and $x(0)=0$, then $x\equiv 0$.*

Then, $(A_n)_{n\in\mathbb{Z}^-}$ admits NED.

Remark 2.15. *If operators A_m are invertible, then (28) can be replaced by the requirement that $\tilde{U}_{\gamma_1,\mathbb{Z}^-}(0)=\tilde{U}_{\gamma_2,\mathbb{Z}^-}(0)$. Moreover, in this case the assumption (d) can be eliminated.*

Finally, we discuss the case of dichotomies on $\mathbb{Z}$.

Theorem 2.16. *Assume that there are $0<\delta_i\leq\gamma_i$, $i\in\{1,2\}$ with $\gamma_1<1$ and $\gamma_2>1$ such that:*

(1) *the pairs $(\mathcal{L}^\infty_{\delta_i}(\mathbb{Z}),\mathcal{L}^\infty_{\gamma_i}(\mathbb{Z}))$ are* properly *admissible for $i=1,2$;*

(2) *we have that*

$$U_{\gamma_1,\mathbb{Z}^+}(m)=U_{\gamma_2,\mathbb{Z}^+}(m) \quad \textit{for } m\in\mathbb{Z}^+$$

and

$$\tilde{U}_{\gamma_1,\mathbb{Z}^-}(m)=\tilde{U}_{\gamma_2,\mathbb{Z}^-}(m) \quad \textit{for } m\in\mathbb{Z}^-;$$

(3) *$U_{\gamma_1,\mathbb{Z}^+}(0)$ and $\tilde{U}_{\gamma_1,\mathbb{Z}^-}(0)$ are closed.*

Then, $(A_n)_{n\in\mathbb{Z}}$ admits NED.

Proof. Firstly, it follows easily from the first assumption of the theorem that the pairs $(\mathcal{L}^\infty_{\delta_i}(\mathbb{Z}^+),\mathcal{L}^\infty_{\gamma_i}(\mathbb{Z}^+))$ and $(\mathcal{L}^\infty_{\delta_i}(\mathbb{Z}^-),\mathcal{L}^\infty_{\gamma_i}(\mathbb{Z}^-))$, $i=1,2$, are admissible. Moreover, by arguing exactly as in the proof of [27, Theorem 2.2.] we obtain that

$$X=U_{\gamma_1,\mathbb{Z}^+}(0)\oplus\tilde{U}_{\gamma_1,\mathbb{Z}^-}(0)=U_{\gamma_2,\mathbb{Z}^+}(0)\oplus\tilde{U}_{\gamma_2,\mathbb{Z}^-}(0).$$

Thus, $U_{\gamma_1,\mathbb{Z}^+}(0)$ and $\tilde{U}_{\gamma_1,\mathbb{Z}^-}(0)$ are complemented. Furthermore, by the *proper* admissibility of $(\mathcal{L}^\infty_{\delta_1}(\mathbb{Z}),\mathcal{L}^\infty_{\gamma_1}(\mathbb{Z}))$ we get that the assumption (*d*) of Theorem 2.14 is also satisfied. Consequently, by Theorems 2.7 and 2.14 we have that $(A_n)_{n\in\mathbb{Z}^+}$ and $(A_n)_{n\in\mathbb{Z}^-}$ admit NED with projections P^+_n, $n\in\mathbb{Z}^+$ and P^-_n, $n\in\mathbb{Z}^-$, respectively. Therefore, there exist $D\geq 1$, $\alpha\in(0,1)$ and $\varepsilon\geq 1$ such that

$$|\Phi(m,n)P^+_n|\leq D\alpha^{m-n}\varepsilon^n \quad m\geq n\geq 0, \tag{29}$$

$$|\Phi(m,n)(Id-P^+_n)|\leq D\alpha^{n-m}\varepsilon^n \quad 0\leq m<n, \tag{30}$$

$$|\Phi(m,n)P^-_n|\leq D\alpha^{m-n}\varepsilon^{|n|} \quad 0\geq m\geq n, \tag{31}$$

and

$$|\Phi(m,n)(Id-P^-_n)|\leq D\alpha^{n-m}\varepsilon^{|n|} \quad m<n\leq 0. \tag{32}$$

Moreover, we can choose P^+_0 and P^-_0 so that

$$ImaP^+_0=ImaP^-_0=U_{\gamma_1,\mathbb{Z}^+}(0) \quad \text{and} \quad KerP^+_0=KerP^-_0=\tilde{U}_{\gamma_1,\mathbb{Z}^-}(0).$$

Hence, $P_0^+ = P_0^-$. Set

$$P_n = \begin{cases} P_n^+ & n \geq 0; \\ P_n^- & n < 0. \end{cases}$$

Then, $A_n P_n = P_{n+1} A_n$ for each $n \in \mathbb{Z}$. Moreover, $A_n|_{KerP_n}$ is invertible for every $n \in \mathbb{Z}$. For $m > 0 > n$ we have using (29) and (31) that

$$|\Phi(m,n)P_n| = |\Phi(m,0)P_0^+ \Phi(0,n)P_n^-| \leq D\alpha^m |\Phi(0,n)P_n^-| \leq D^2 \alpha^{m-n} \varepsilon^{|n|}.$$

This together with (29) and (31) yields that

$$|\Phi(m,n)P_n| \leq D^2 \alpha^{m-n} \varepsilon^{|n|}, \quad \text{for } m,n \in \mathbb{Z} \text{ such that } m \geq n.$$

Similarly, (30) and (32) imply that

$$|\Phi(m,n)(Id - P_n)| \leq D^2 \alpha^{n-m} \varepsilon^{|n|}, \quad \text{for } m,n \in \mathbb{Z} \text{ such that } m \leq n.$$

Hence, we obtain the conclusion of the theorem. □

Remark 2.17. *We observe that, since we improve the main results of* [27] *to the case of* infinite dimensional *and* noninvertible *dynamics and shorten their proofs via different methods, we also improve the previous literature* [2, 4, 5, 31, 30] *in the sense that neither bounded growth nor Lyapunov norms are necessary in this paper.*

References

[1] L. Backes, D. Dragičević, Periodic approximation of exceptional Lyapunov exponents for semi-invertible operator cocycles, *Ann. Acad. Sci. Fenn. Math,* 44 (2019), 183–209.

[2] L. Barreira, D. Dragičević and C. Valls, Admissibility and nonuniform hyperbolicity, *Adv. Nonlinear Stud.* 14 (2014), 791–811.

[3] L. Barreira, D. Dragičević, C. Valls, Strong and weak (L^p, L^q)-admissibility, *Bull. Sci. Math.,* 138 (2014) 721–741.

[4] L. Barreira, D. Dragičević, C. Valls, Admissibility on the half line for evolution families, *J. Anal. Math.*, 132 (2017) 157–176.

[5] L. Barreira, D. Dragičević and C. Valls, *Admissibility and Hyperbolicity*, SpringerBriefs Math., Springer, Cham, 2018.

[6] L. Barreira and Ya. Pesin, *Nonuniform hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents.* Enc. of Mathematics and its Applications 115, Cambridge University Press, 2007

[7] L. Barreira and C. Valls, *Stability of nonautonomous differential equations*, Lecture Notes in Math., 1926 Springer, Berlin, 2008. xiv+285 pp.

[8] C.V. Coffman, J.J. Schäffer, Dichotomies for linear difference equations, *Math. Ann,* 172 (1967) 139–166.

[9] W.A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Math., vol. 629, Springer, Berlin, 1978.

[10] J.L. Dalec'kiĭ, M.G. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Transl. Math. Monogr., vol. 43, American Mathematical Soc., Providence, RI, 1974.

[11] D. Dragičević, W. Zhang, L. Zhou, Admissibility and nonuniform exponential dichotomies, *J. Differential Equations,* 326 (2022) 201–226.

[12] D. Dragičević, W. Zhang, L. Zhou, Measurable weighted shadowing for random dynamical systems on Banach spaces, *J. Differential Equations,* 392 (2024), 364–386.

[13] D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Lecture Notes in Math., vol. 840, Springer, Berlin, 1981.

[14] N. T. Huy, Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line, *J. Funct. Anal.* 235 (2006), 330–354.

[15] N. T. Huy, N.V. Minh, Exponential dichotomy of difference equations and applications to evolution equations on the half-line, *Comput. Math. Appl.,* 42 (2001) 301–311.

[16] Y. Latushkin, T. Raudolph, R. Schnaubelt, Exponential dichotomy and mild solutions of nonautonomous equations in Banach spaces, *J. Dynam. Differential Equations*, 10 (1998) 489–510.

[17] T. Li, Die stabilitätsfrage bei differenzengleichungen, *Acta Mathematica*, 63 (1934) 99–141.

[18] J.L. Massera and J.J. Schäffer, Linear differential equations and functional analysis I, *Ann. Math.* 67 (1958) 517–573.

[19] J.L. Massera and J.J. Schäffer, *Linear Differential Equations and Function Spaces,* Academic Press, New York, 1966.

[20] M. Megan, B. Sasu and A. L. Sasu, On nonuniform exponential dichotomy of evolution operators in Banach spaces, *Integral Equations Operator Theory*, 44 (2002), 71–78.

[21] M. Megan, A.L. Sasu, B. Sasu, Discrete admissibility and exponential dichotomy for evolution families, *Discrete Contin. Dyn. Syst.*, 9 (2003) 383–397.

[22] M. Megan, A.L. Sasu, B. Sasu, Uniform exponential dichotomy and admissibility for linear skew-product semiflows, *Recent Advances in Operator Theory, Operator Algebras, and their Applications,* 185–195, Springer, 2004.

[23] O. Perron, Die stabilitätsfrage bei differenzengleichungen, *Math. Z*, 32 (1930) 703–728.

[24] C. Preda, A discrete Perron–Ta Li type theorem for the dichotomy of evolution operators, *J. Math. Anal. Appl.,* 332 (2007) 727–734.

[25] P. Preda, M. Megan, Nonuniform dichotomy of evolutionary processes in Banach spaces, *Bull. Austral. Math. Soc.* 27 (1983), 31–52.

[26] P. Preda, A. Pogan, C. Preda, Schäffer spaces and exponential dichotomy for evolutionary processes, *J. Differential Equations,* 230 (2006) 378–391.

[27] M. Wu and Y. Xia, Admissibility and nonuniform exponential dichotomies for difference equations without bounded growth or Lyapunov norms, *Proc. Amer. Math. Soc.* 151 (2023), 4389–4403.

[28] A. L. Sasu, M. G. Babuţia, B. Sasu, Admissibility and nonuniform exponential dichotomy on the half-line, *Bull. Sci. Math.* 137 (2013), 466–484.

[29] B. Sasu, A.L. Sasu, Exponential dichotomy and (ℓ^p, ℓ^q)-admissibility on the half-line, *J. Math. Anal. Appl.,* 316 (2006) 397–408.

[30] L. Zhou, W. Zhang, Admissibility and roughness of nonuniform exponential dichotomies for difference equations, *J. Funct. Anal.*, 271 (2016) 1087–1129.

[31] L. Zhou, K. Lu, W. Zhang, Equivalences between nonuniform exponential dichotomy and admissibility, *J. Differential Equations*, 262 (2017) 682–747.

Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, CEP 91509-900, Porto Alegre, RS, Brazil.
Email address, Lucas Backes: `lucas.backes@ufrgs.br`

Department of Mathematics, University of Rijeka, Croatia
Email address, Davor Dragičević: `ddragicevic@math.uniri.hr`

School of Mathematical Science, Zhejiang Normal University, Jinhua 321004, People's Republic of China
Email address, Yonghui Xia: `yhxia@zjnu.cn`