RG 2015/2

$\boxed{Q1}$ (a) $y' = -xy^2 \Leftrightarrow -\frac{dy}{y^2} = x\,dx \Leftrightarrow \frac{1}{y} = \frac{x^2}{2} + \frac{c}{2}, \; c \in \mathbb{R}$

$y = \frac{2}{c+x^2} \qquad y(1) = 1/2 \Leftrightarrow \frac{1}{2} = \frac{2}{c+1} \Rightarrow c = 3$

$\Rightarrow \; y(x) = \frac{2}{3+x^2}, \; x \geq 1$ é sol do PVI

(b) $y' = -\frac{e^x + y}{2 + x + ye^y}$ re-escreve $(2+x+ye^y)dy = -(e^x+y)dx$

e na forma $M dx + N dy = 0$ onde $\begin{cases} M = e^x + y \\ N = 2 + x + ye^y \end{cases}$

compatibilidade : $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$ EQ É EXATA

$f_x = e^x + y \Rightarrow f = e^x + xy + g(y)$

$f_y = x + g'(y) = 2 + x + ye^y \Rightarrow g' = 2 + ye^y$

$\Rightarrow g = 2y + ye^y - e^y \Rightarrow f = e^x + xy + 2y + ye^y - e^y$

solução geral $\quad e^x + xy + 2y + ye^y - e^y = C, \; C \in \mathbb{R}$

$y(0) = 1$ IMPLICA $\quad 1 + 0 + 2 + 1 - 1 = C \Rightarrow C = 3$

solução do PVI na forma implícita

$$e^x + (x+2)y + (y-1)e^y = 3$$

(c) re-escreve $\frac{y'}{y} = \frac{1-x}{x^2} = \frac{1}{x^2} - \frac{1}{x} \Rightarrow \ln|y| = -\frac{1}{x} - \ln|x| + C, \; C \in \mathbb{R}$

$\Rightarrow y(x) = \exp\left(C - \frac{1}{x} - \ln|x|\right) = C_1 |x|^{-1} e^{-1/x}$

$y(1) = 1 \Rightarrow 1 = C_1 \cdot 1 \cdot e^{-1} \Rightarrow C_1 = e$

sol do PVI $\quad y(x) = \frac{e \cdot e^{-1/x}}{x}, \; x \geq 1$

(d) re-escreve $xy^2 dy = (y^3 - x^3)dx$ $\quad M dx + N dy = 0$

onde $M = x^3 - y^3$ $\quad M_y = -3y^2$ $\quad$ NÃO É EXATA

$N = xy^2$ $\quad N_x = +y^2$

MAS $\dfrac{M_y - N_x}{N} = \dfrac{-3y^2 - y^2}{xy^2} = -\dfrac{4}{x}$

FATOR INTEGRANTE $\mu = \exp\left(-\int \frac{4}{x} dx\right) = e^{-4\ln|x|} = x^{-4}$

IMPLICA $\dfrac{x^3 - y^3}{x^4} dx + \dfrac{xy^2}{x^4} dy = 0$

$f_x = \dfrac{x^3 - y^3}{x^4} = \dfrac{1}{x} - \dfrac{y^3}{x^4} \Rightarrow f = \ln|x| + \dfrac{x^{-3}y^3}{3} + g(y)$

$f_y = +x^{-3}y^2 = N \Rightarrow g = 0$

$\Rightarrow f = \ln|x| + \dfrac{y^3}{3x^3}$

solução geral $\ln|x| + \dfrac{y^3}{3x^3} = C,\ C \in \mathbb{R}$

$\Rightarrow y^3 = 3x^3 (C - \ln|x|)$

$y(1) = 2 \Rightarrow 8 = 3(1)(C - 0) \Rightarrow C = 8/3$

$\Rightarrow y^3 = 3x^3 (8/3 - \ln|x|) = x^3 (8 - 3\ln|x|)$

solução do PVI $\quad y = x(8 - 3\ln|x|)^{1/3},\ x \geq 1$

$\boxed{Q2}$ (a) homog: $u'' - 16u = 0$ $\quad \alpha^2 - 16 = 0,\ \alpha = \pm 4$

$u(x) = Ae^{4x} + Be^{-4x},\ A, B \in \mathbb{R}$

(b) procura $y_p = Cxe^{4x}$ (pois e^{4x} é sol homog)

$y_p' = Ce^{4x} + 4Cxe^{4x}$

$y_p'' = 8Ce^{4x} + 16Cxe^{4x}$

$8Ce^{4x} + 16Cxe^{4x} - 16Cxe^{4x} = 2e^{4x} \Rightarrow C = 1/4$

solução geral

$y = Ae^{4x} + Be^{-4x} + \dfrac{xe^{4x}}{4},\ A, B \in \mathbb{R}$

$\boxed{Q3}$ $\dfrac{dX}{dt} = AX + F(t) \quad A = \begin{bmatrix} 1 & -4 \\ -6 & -1 \end{bmatrix}, \; F = \begin{bmatrix} -7 \\ 2 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \end{bmatrix}$

autovalores da matriz A : $(1-\lambda)(-1-\lambda) - 24 = 0$

$\lambda^2 - 1 - 24 = 0 \Rightarrow \lambda = \pm 5$

$\lambda = -5$: $x_1 - 4x_2 = -5x_1 \Leftrightarrow -4x_2 = -6x_1 \Leftrightarrow 2x_2 = 3x_1$

$K_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gera subespaço de autovetores

$\lambda = 5$: $x_1 - 4x_2 = 5x_1 \Leftrightarrow -4x_2 = 4x_1 \Leftrightarrow x_2 = -x_1$

$K_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ gera subespaço de autovetores

solução particular constante:

$\begin{cases} x - 4y = 7 \\ -6x - y = -2 \end{cases} \Rightarrow \begin{matrix} x - 4y = 7 \\ 6x + y = 2 \end{matrix} \Rightarrow \begin{matrix} x - 4y = 7 \\ 24x + 4y = 8 \end{matrix}$

$25x = 15 \Rightarrow x = 3/5$

$y = 2 - 6x = 2 - 6(3/5) = -8/5$

$\Rightarrow X(t) = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + \dfrac{1}{5} \begin{bmatrix} 3 \\ -8 \end{bmatrix}, \; c_1, c_2 \in \mathbb{R}$

$\boxed{Q4}$ (a) $Y = \mathcal{L}\{y\}$ $\quad sY - 2 - Y = \dfrac{2}{s-5}$

$\Rightarrow Y(s-1) = 2 + \dfrac{2}{s-5} \Rightarrow Y = \dfrac{2}{s-1} + \dfrac{2}{(s-1)(s-5)}$

DFP: $\dfrac{2}{(s-1)(s-5)} = \dfrac{A}{s-1} + \dfrac{B}{s-5} = \dfrac{A(s-5) + B(s-1)}{(s-1)(s-5)}$

$s = 1 : 2 = A(-4) \Rightarrow A = -1/2$

$s = 5 : 2 = B(4) \Rightarrow B = 1/2$

$y(t) = \mathcal{L}^{-1}\left\{ \dfrac{2}{s-1} + \dfrac{-1/2}{s-1} + \dfrac{1/2}{s-5} \right\} = \mathcal{L}^{-1}\left\{ \dfrac{3/2}{s-1} + \dfrac{1/2}{s-5} \right\}$

$y(t) = \dfrac{3}{2} e^{t} + \dfrac{1}{2} e^{5t}$ solução do PVI

(b) $sY-(-2)-Y = \dfrac{d^2}{ds^2}\left[\dfrac{1}{s-1}\right] \Rightarrow (s-1)Y = -2 + \dfrac{2}{(s-1)^3}$

$$\Rightarrow Y(s) = \frac{-2}{s-1} + \frac{2}{(s-1)^4}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s-1} + \frac{2}{(s-1)^4}\right\} = -2e^t + \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}$$

$$= -2e^t + \frac{t^3e^t}{3}$$

(c) $s^2Y - s(0) - 0 + 4Y = \dfrac{e^{-2s}}{s} \Rightarrow Y = \dfrac{e^{-2s}}{s(s^2+4)}$

fórmula $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\} = \dfrac{\operatorname{sen}(2t)}{2}$

integral $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\} = \left[\dfrac{-\cos(2u)}{4}\right]_0^t = \dfrac{1-\cos(2t)}{4}$

fórmula $y = \mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s^2+4)}\right\} = \dfrac{1-\cos(2(t-2))}{4}\, u(t-2)$ //

alternativamente

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{A(s^2+4)+s(Bs+C)}{s(s^2+4)}$$

$s=0$: $1 = A(4) \Rightarrow A = 1/4$

$s=1$: $1 = A(5) + 1(B+C) \Rightarrow B+C = -1/4$

$s=-1$: $1 = A(5) + (-1)(-B+C) \Rightarrow B-C = -1/4$

$\Rightarrow 2B = -1/2 \Rightarrow B = -1/4 \Rightarrow C = 0$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/4}{s} - \frac{s/4}{s^2+4}\right\}$$

$$= \frac{1}{4} - \frac{1}{4}\cos(2t)$$

e da mesma forma

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+4)}\right\} = \frac{1-\cos(2(t-2))}{4}\cdot u(t-2)$$

(d) $s^2Y - s(0) - (2) - 4(sY - 0) + 3Y = \frac{1}{s^2}$

$(s^2 - 4s + 3)Y = 2 + \frac{1}{s^2}$ MAS $(s^2 - 4s + 3) = (s-3)(s-1)$

$$\Rightarrow Y(s) = \frac{2}{(s-3)(s-1)} + \frac{1}{s^2(s-3)(s-1)}$$

DFP: $\frac{2}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1} = \frac{A(s-1) + B(s-3)}{(s-3)(s-1)}$

$2 = A(s-1) + B(s-3) = (A+B)s - (A+3B)$

$\begin{cases} A+B=0 \\ A+3B=2 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases} \Rightarrow \frac{2}{(s-1)(s-3)} = \frac{1}{s-3} - \frac{1}{s-1}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2}{(s-3)(s-1)}\right\} = e^{3t} - e^{t}$$

aplicando a fórmula da integral:

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s-3)(s-1)}\right\} = \left[\frac{e^{3u}}{3} - e^{u}\right]_0^t = \frac{e^{3t}}{3} - e^{t} + \frac{2}{3}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2(s-3)(s-1)}\right\} = \left[\frac{e^{3u}}{9} - e^{u} + \frac{2u}{3}\right]_0^t$$

$$= \frac{e^{3t}}{9} - e^{t} + \frac{2t}{3} + \frac{8}{9}$$

Finalmente

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s-3)(s-1)} + \frac{1}{s^2(s-3)(s-1)}\right\} =$$

$$= e^{3t} - e^{t} + \frac{1}{2}\left[\frac{e^{3t}}{9} - e^{t} + \frac{2t}{3} + \frac{8}{9}\right]$$

$$= \frac{19e^{3t}}{18} - \frac{3}{2}e^{t} + \frac{t}{3} + \frac{4}{9}$$