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A SEMIOTIC PERSPECTIVE OF MATHEMATICAL ACTIVITY: THE CASE OF NUMBER

ABSTRACT. A semiotic perspective on mathematical activity provides a way of conceptualizing the teaching and learning of mathematics that transcends and encompasses both psychological perspectives focussing exclusively on mental structures and functions, and performance-focussed perspectives concerned only with student' behaviours. Instead it considers the personal appropriation of signs and the underlying meaning structures embodying relationships between signs. It is concerned with patterns of sign use and sign production, including individual creativity in sign use, and the underlying social rules and contexts of sign use. It is based on the concept of a semiotic system, comprising signs, rules of sign production, and an underpinning meaning structure. This theorisation is applied to the learning of number, from counting to calculation. Historical, foundational and developmental (i.e., learning) perspectives are explored and contrasted. It is argued that in each of these domains, the dominant significant activity concerns the production of sequences of signs.

KEY WORDS: development of number understanding, foundations of number, history of number, mathematical activity, mathematics, mathematics education, mathematical signs, number, numerical calculation, semiotics, teaching and learning mathematics

INTRODUCTION

In proposing a semiotic view of mathematics education as an additional way of conceptualising the teaching and learning of mathematics, a rationale is called for. Some justification needs to be given for the proposal of yet another perspective, given the range of theories already utilised in mathematics education. The justification is multiple, and is based on the role of semiotics as the study of signs encompassing all aspects of human sign making, reading and interpretation, across the multiple contexts of sign usage. Mathematics is an area of human endeavour and knowledge that is known above all else for its unique range of signs and sign-based activity. So it seems appropriate to apply the science of signs to mathematics. Likewise in schooling, learners meet a whole new range of signs and symbolising functions in mathematics. So again it seems appropriate to adopt a sign-orientated perspective from which to examine school mathematics.

This is not an isolated view or application of this approach. A growing number of scholars are applying the tools of semiotics to mathematics and mathematics education, the work of Anderson et al. (2003), Chapman (1992), Ernest (1997), 1998a, 2003; Kirshner and Whitson (1997), Morgan

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(1998), Pimm (1995), Radford (1998) and Rotman (1987, 1993), and others testifies, as does the existence of this collection. In addition, a semiotic approach follows naturally from some of the recent Vygotskian, Activity Theoretic, socio-cultural and social constructivist theorisations of mathematics and its teaching and learning. So this newer approach holds promise and is already bearing fruit. Ultimately, the value of such a theorisation must be judged on the new insights it provides and on any enhancement of school practices in mathematics to which it leads. Only the beginnings of such contributions are offered here.

A semiotic perspective of mathematical activity provides a way of conceptualising the teaching and learning of mathematics driven by a primary focus on signs and sign use.[1] In providing this viewpoint it offers an alternative to psychological perspectives that focus exclusively on mental structures and functions. It also rejects any straightforwardly performance or assessment focussed perspectives concerned only with student behaviours. Instead it offers a novel synthesis that encompasses but also transcends these two types of perspective, driven by a primary focus on signs and sign use in mathematics. Beyond the traditional psychological concentration on mental structures and functions 'inside' an individual it considers the personal appropriation of signs by persons within their social contexts of learning and signing. Beyond behavioural performance this viewpoint also concerns patterns of sign use and production, including individual creativity in sign use, and the underlying social rules, meanings and contexts of sign use as internalised and deployed by individuals. Thus a semiotic approach draws together the individual and social dimensions of mathematical activity as well as the private and public dimensions. These dichotomous pairs of ideas are understood as mutually dependent and constitutive aspects of the teaching and learning of mathematics, rather than as standing in relations of mutual exclusion and opposition. Individual learning is initiated by participation in and partaking of the social. Likewise, the effective public use of signs rests on the private construction of meaning. This is perhaps the main justification for the adoption of a semiotic perspective on mathematics and on the teaching and learning of mathematics. It transcends the limits of purely cognitive and behavioural psychological approaches through providing a basic and natural unit of intelligent action, the sign.

A semiotic perspective also transcends the traditional subjective-objective dichotomy. For signs are intersubjective, and thus provide both the basis for subjective meaning construction, as well as the basis for shared human knowledge, which as I have argued elsewhere, is what is taken for objective knowledge (Ernest, 1991, 1998a). This has deep epistemological implications. But it also opens up new ways of exploring the teaching and learning of mathematics.

The primary focus in a semiotic perspective is on communicative activity in mathematics utilizing signs. This involves both sign reception and comprehension via listening and reading, and sign production via speaking and writing or sketching. While these two directions of sign communication are conceptually distinct, in practice these types of activity overlap and are mutually shaping in conversations, i.e., semiotic exchanges between persons within a social context. Sign production or utterance is primarily an agentic act and often has a creative aspect. For the speaker has to choose and construct texts to utter, i.e., to speak or write, on the basis of their appropriated and learned repertoire of signs. In so doing, speakers are taking risks in exposing themselves to external correction and evaluation against the rules of appropriate utterances. However, except in pathological situations, such as the speaker having a history of stigmatised failure (which is not uncommon in mathematics, Buxton, 1981; Maxwell, 1989), such risks are not usually perceived as threatening, or even as risky. For there is a joy in utilizing ones powers to participate in social activities, to articulate one's response or self-initiated meanings in some text. (Here, as is widespread in semiotics, 'text' denotes more than a piece of writing. A text is a compound sign made up of constituent signs, and can be uttered or offered in a conversation in many ways. It may be spoken, written, drawn, represented electronically and may include gestures, letters, mathematical symbols, diagrams, tables, etc., or some selection or combination of these modes.)

The perspective I wish to sketch here focuses on mathematical sign systems and the mathematical content, skills and capacities developed and elaborated during the educational process. However, from a semiotic perspective it is always the case that signs and sign use can only be understood as part of more complex systems. First, all sign use is socially located and is part of social and historical practice. In Wittgensteinian (1953) terms sign use comprises 'language games' embedded in social 'forms of life' (Ernest, 1998a). Second, signs are never used in isolation, for signs are always manifested as part of semiotic systems, with reference implicitly or explicitly, to other signs. The term semiotic system is here used to comprise three necessary components. First, there is a set of signs, each of which might possibly be uttered, spoken, written, drawn, or encoded electronically. Second, there is a set of rules of sign production, for producing or uttering both atomic (single) and molecular (compound) signs. These rules concern much more than the definition and determinants of a well-formed, i.e., grammatically correct, sign. They also concern the sequencing of signs in conversation, i.e., what sign utterances may legitimately follow on from prior signs in given social contexts. In mathematics this includes rules that legitimate certain text transformations, e.g., 'canceling', the common division of numerator and denominator, in fractions. Third, there is a set of relationships between the signs and their meanings embodied in an underlying meaning structure.

This three component model is a simplified one which foregrounds the system of signs in order to facilitate the explication of mathematical structures without indicating the irreducible and ever-present social dimensions, such as the roles of speakers, social functions, and so on. Halliday (1978) and Halliday and Martin (1993), for example, provide a powerful theoretical treatment of these missing dimensions that has been used in other semiotic and linguistic studies in mathematics education (e.g., Chapman, 1992; Morgan, 1998; Pimm, 1995). The social and historical embedding of semiotic systems concerns both their structural dimension, Saussure's *langue*, and in their functional role, *parole* (Saussure, 1916). The evolution of semiotic systems can be examined historically in terms of both of these dimensions. While theoretically separable, the two dimensions are woven together in socio-historical practice, where the underlying given is people's actual use of signs in a variety of social settings and epochs.

Although the structural dimension of semiotic systems is, in the main, a theoretical abstraction, the unusual occurs in mathematics, namely, this dimension is also manifested in practice. Mathematical theories representing the structural part of mathematics are part of the functional *parole*, rather than being solely *post hoc* theoretical constructions within the *langue*. The construction and explicit formulation of mathematical theories has been an important and recognised part of mathematical practice ever since Euclid's articulation of the *Elements* of geometry. To put it another way, metamathematics is a part of mathematics.

To a mathematician it might be tempting to represent a semiotic system as defined above in the style of a mathematical structure, as an ordered triple $\langle S, R, M \rangle$, where S is a set of signs, R is a set of rules, and M the underlying meaning structure. However, this temptation should be resisted for it gives a misleading sense of precision and definiteness to the idea of a semiotic system. R is at best a fuzzy set and the potential members of M can never be made explicit, let alone represented as definite and well-defined set.

Although mathematical theories are manifested in mathematical practice, they cannot make semiotic systems fully explicit. For while it is difficult to render all of the rules of sign production explicit in a formal theory, theoretically it might be done. The more abstract and formal the theory, the further removed from actual social use, the easier it is. But even where this is accomplished, if indeed it can be accomplished at all, the underlying meaning structure cannot be so represented. Just as in any language, there is an irreducible tacit knowledge element (Ernest, 1999; Polanyi, 1958).

Despite the pull towards abstraction, semiotic systems cannot be severed from human understanding and use. They lose meaning once isolated as purely structural systems.

Explicitly formulated mathematical theories are not conducive to students' learning of mathematics for the reason that they do not and cannot communicate the meanings underlying the signs and rules (in addition to the difficulties associated with the abstraction and generality of their formulation). Philosophers of mathematics, especially formalists like David Hilbert and logicists like Bertrand Russell have aspired to expressing mathematical knowledge completely in formal mathematical theories. However, one of Imre Lakatos' (1976, 1978) enduring contributions is his stubborn insistence, and compelling reasoning, that formal mathematical theories always have a 'shadow', the informal realm of meaning roughly corresponding to the meaning structure in a semiotic system, as defined above. Any formal moves to sever this realm of meaning not only render mathematical theories meaningless to learners, but also cut off the sources of intuition and renewal from research mathematicians. Some meanings can be expressed textually (i.e., using signs) but there must always be a residual rump of unexpressed meaning. By definition, and on pain of circularity, the production of new signifiers always implicates new signifieds. Or to put it another way, understanding can never be abstracted out of its living human contexts. There are also more technical problems in the foundations of mathematics resulting from attempts to replace meaningful informal theories by rigorous axiomatic formulations. For example, as Gödel (1931) demonstrated, no formal theory can accurately and consistently express the content of an informal theory as complex as arithmetic. Further, as Skolem showed, every formal (first-order) theory for arithmetic, set theory, etc., always admits large numbers of unintended nonstandard domains as legitimate interpretations (Machover, 1996).

In an educational context, when semiotic systems are presented only one component is made explicit, namely, the set of signs. Even in this case only some signs are explicitly presented. All atomic signs, the base symbols, e.g., single digit numerals, and later operator symbols such as $+$, $-$, $=$, will normally be explicitly displayed, but typically the set of compound signs, e.g., multi-digit numerals, is inexhaustible, and new signs are continuously being produced. As the product of creative use, these signs cannot all be anticipated. The rules of sign production are in most cases implicit, and are acquired by 'case law' by novice users of a semiotic system. That is, they are learned by seeing their applications and uses in social practice. Once a semiotic system is fully developed historically or mathematically the rules might be made more explicit. But from a developmental, i.e., a learner's perspective, a semiotic system is mostly in the process of unfolding and emerging in classroom practice. In this, and indeed in any other context, the set of underlying meanings can never be rendered explicitly.

From a semiotic perspective, none of this comes as a surprise. Semiotic systems involve signs, rules of sign use and production, and underlying meanings. All of these depend on social practices, and human beings as quintessentially sign using and meaning making creatures can never be eliminated from the picture, even if for some purposes we foreground signs and rules and background people and meanings.

Historical and cultural developments in mathematics give rise to the semiotic systems that provide the underlying structure to the learning environments planned for students, namely, mathematical theories. These are selected from, reduced, adapted and recontextualised as the systematic topic domains of school mathematics (Dowling, 1988). The relationship between the topic areas and structures of the planned and taught mathematics curriculum and the learned curriculum as appropriated by students embodies the well known dictum of Vygotsky (1978: 128) "Every function in the child's cultural development appears twice, on two levels. First, on the social and later on the psychological level; first between people as an interpsychological category, and then inside the child as an intrapsychological category." Likewise, the historical and mathematical developments in semiotic systems, when suitably recontextualised and reconfigured, become the subject matter for the teaching and learning of mathematics, which individual students master. In simplified terms, the social becomes personal, and the public becomes private.

Figure 1 below illustrates the stages in such transformations of semiotic systems in school mathematics. It begins where, for educational purposes,

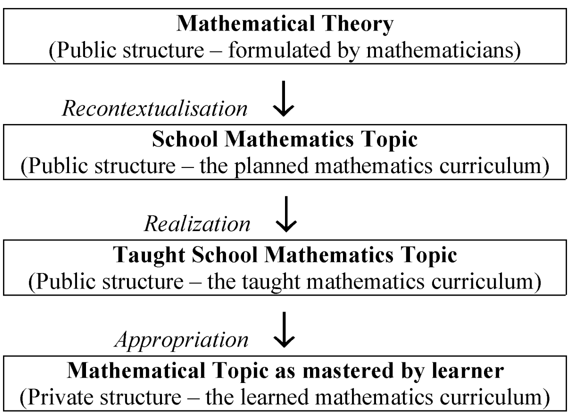

Figure 1. The transformations of semiotic systems in school mathematics.

a selection is made of some target mathematical theory from the totality of publicly available mathematical knowledge. Given this selection, elements of a mathematical theory are recontextualised into a school mathematics topic. This is transformed through the teacher's presentation and appropriated by a student into their own mastery of the associated semiotic system. The planned becomes the taught becomes the learned mathematics curriculum (Robitaille and Garden, 1989).

Even in terms of this very simple model, there are a number of complexities to be remarked upon. First of all, for any particular theory or area of mathematics, there is no fixed or unique mathematical theory formulation as a source. There are always multiple formulations by different mathematicians and groups of mathematicians constructed and published at different times. Furthermore, most school mathematics topics are no longer a part of academic (university) mathematics and thus figure in no contemporary academic textbooks. Examples include whole number, fractional and decimal operations; ratio; percentages; trigonometry; elementary algebra. These tend to be regarded as trivial prerequisites to more advanced mathematical expositions. The most common sources of exposition of these mathematical topic areas are textbooks designed for use in schools or teacher education. Such textbooks are not always reliable, as unlike papers in mathematical journals, there is no guarantee that they have been checked scrupulously or with sufficient rigour by experts.

Although not mentioned in contemporary advanced academic textbooks, elementary topics like those listed above are treated in historical textbooks, such as *The Treviso Arithmetic*, R. Recorde's *Arithmetic*, S. Stevin's *Decimal Fractions* (see Smith, 1959), and more recent (but still historic) publications. At the time of publication these historical sources were both advanced academic treatises for scholars, as well as teaching texts. Even today most published volumes of academic mathematics (as opposed to journal papers) are advanced exposition for educational purposes. Nevertheless, virtually none of current school mathematics has been the subject of leading edge mathematical research in the past century or two, except for foundational purposes unsuitable for teaching.

Second, given the choice of any particular formulation of a mathematical theory, there is no limit to the number of ways of recontextualising it as a school mathematics topic in the planned mathematics curriculum. Third, every realisation of the planned mathematics curriculum as the taught curriculum is potentially unique. For the presentation of this public structure has elements of performance in it, in the live interactions between teacher and students, and these can never be repeated identically, even if the same textbook selections, chalkboard notes, worksheets, etc., are used again. To some extent this mirrors the relationship between *langue*, corresponding to the planned topic structure, and its manifestation in *parole*, corresponding to the sequence of utterances by the teacher, including all representations used in classroom teaching. Fourth, there is learned mathematics curriculum, which because of the idiosyncratic nature of the process of appropriation and the underlying personal construction of meaning by learners, also results in widely varying outcomes.

One of the dangers of a simple model like the above is that it might appear to correspond to an educational and epistemological misconception about teaching and the transmission of knowledge. As is now widely accepted, knowledge is not analogous to a material entity that can be transmitted from one person to another, or delivered like a book or computer disc. Signs and rules can, in the extreme case, be 'transmitted', i.e., directly communicated, although this is not an effective teaching strategy. Seeger and Steinbring (1990) directly address the problems of the transmission metaphor in education and underscore its weaknesses. In addition, there is a large literature on the problems of teaching solely by exposition and direct instruction, and learning solely by rote and the practice and reinforcement of skills (Ausubel, 1968; Bell et al., 1983; Skemp, 1976). The key issue is that meanings have to be constructed by each human being in turn, and the making of meanings by a person always draws upon their active (if unconscious) mobilisation of existing elements of meaning and understanding. These are enduring insights offered by cognitive, constructivist as well as social theories of learning, irrespective of any controversies between them (Ernest, 1994b; Steffe and Gale, 1995).

In contrast to simplistic transmission approaches, great pedagogical knowledge and skill is needed for the successful teaching of a mathematical topic. It needs to be presented within a supportive social context using a variety of representations and tasks with sufficient redundancy so that rules and relationships are inferred and meanings constructed and elaborated. This normally takes place during an asymmetric dialogue between a teacher and a class of students in which burgeoning capabilities are elicited and assessed. For inevitably a student's understanding and construction of meaning can only be assessed indirectly through their public performance. Using the signs of mathematics, in a variety of tasks and settings, is naturally also the main means of acquiring mastery over them for learners too.

Some of the ways in which the teaching of mathematics is much more difficult than the transmission model suggests is illustrated by what might be termed the 'General-Specific paradox'. Typically a teacher wishes the learner to learn some general item of knowledge (sign or rule) that is applicable in multiple and novel situations, such as a mathematical concept, rule, generalised relation, skill or strategy. However if this item of knowledge is presented explicitly as a general statement, often what is learned is precisely this specific statement, such as a definition or descriptive sentence, rather than a human capability. As such it loses its generality and functional power. In order to communicate the content more effectively it needs to be embodied in specific and exemplified terms, typically in a sequence of relatively concrete examples so that the learner can construct and observe the pattern and generalise it, as part of an underlying meaning structure. Thus the paradox is that general understanding is achieved through concrete particulars, and specific responses only may result from general statements. This resembles the Topaze effect (Brousseau, 1997), according to which the more explicitly the teacher states what it is the learner is supposed to learn, the less possible the learning becomes. For the learner is not doing the cognitive work (meaning making) that constitutes learning, but following social cues to provide the required sign (the desired response or answer). This can degenerate into the Jourdain effect (Brousseau, 1997) in which the teacher prematurely accepts a specific and low level sign as evidence of general and higher level understanding.

The model embodied in Figure 1 incorporates a number of complex one-many transformations, as well as manifestations of structures, to a greater or lesser degree of effectiveness, in terms of patterns of utterances. These utterances, including various representations, tasks, conversational exchanges and so on, take place over extended periods of time, which introduces a further dynamic. For just as semiotic systems change and develop over history, so too the semiotic systems mastered by learners change and develop over the course of their learning careers, becoming more elaborated and providing the basis for more complex and abstract systems. It has been found that the private, individual structures, the learned mathematics curriculum as appropriated by students, continue to grow and develop, even without further intentional activity by the teacher (Denvir and Brown, 1986). Although student understanding and mathematical capabilities are described here in terms of the personal construction of private meaning structures, this description is intended to be metaphorical. It is not known how knowledge is privately represented for knowers, and indeed the static, structural language of knowledge structures is purely hypothetical, and in some respects, problematic. The only observable given is student utterances in a variety of modes over a period of time. Once again there is an analogy with the theoretical *langue* versus the realised *parole*. Mental structures are purely hypothetical (and debatable), whereas patterns of utterances and performances are empirically observable. It is the latter that a semiotics approach to mathematics education emphasises.

However, beyond the learners' growing mastery of semiotic systems over time, the planned mathematics curriculum (serving as the basis for the taught and learned curriculum) is also intentionally and systematically changed by the teacher. This is true even within single topic areas, when a new topic area encompasses and subsumes a previously learned topic. Mastering these enlarging semiotic knowledge systems constitutes an important dimension of learning. But such growth and development is not simply a matter of cumulative growth. Nor is it simply a matter of reorganizing the existing elements (signs and rules) or an expanded domain of elements. It is true that as learners near mastery of a particular system, the teacher often extends the system with new signs, relationships, rules or applications. However, sometimes the growth of school semiotic systems involves the negation of existing rules and the change of underlying meaning structures, through the adoption of new rules of sign usage that contradict one or more of the old rules. For example, for a young child mastering elementary calculation, the task 3–4 is impossible. But later it has a determinate answer: $3 - 4 = -1$. Similarly 3 divided by 4 (3/4) is at first an impossible task. Later it is not only a possible task, but 3/4 names the answer to it, i.e., becomes a new kind of semiotic object, a fractional numeral. In early multiplication tasks children learn implicitly or explicitly that "multiplying always makes bigger". Later when the domain of numbers they operate on is expanded to include fractional and decimal numbers (i.e., Rationals), or even just zero, this rule is contradicted.

In these and many comparable cases the rule changes are necessitated by changes in the underlying meaning of the operations. Thus subtraction, initially, is usually understood in enactive or metaphoric terms as resulting from the partitioning of a collection of concrete objects and the removal of one part. Hence $3 - 4$ *is* impossible. Subsequently in learner development subtraction is commonly understood more structurally as the inverse of addition applied to an enlarged and more abstract domain of numbers. Hence since $3 - 3 = 0$, $3 - 4 = -1$. It is very likely that the later more abstract meaning of subtraction cannot be developed without the earlier concrete meaning, so the apparent contradiction is unavoidable.

These, together with more complex changes in the rules that occur, i.e., are imposed, as semiotic systems are extended, and the problems they cause, have been named epistemological obstacles (Bachelard, 1951; Brousseau, 1997; Sierpinska, 1987). For the student has to 'unlearn', that is relinquish something already learned, i.e., part of the meaning structure and any corresponding rules, in order to make further progress. Thus a structural view of semiotic systems can provide only a freeze-frame picture of a growing entity that in its changes almost seems to be alive. Furthermore, not only do individual semiotic systems change and grow. In practice it is difficult to clearly distinguish and demarcate the range of different semiotic systems encountered in school mathematics because of this growth and because of their mutually dependent and constitutive inter-relationships, as elements from one are absorbed into another.

Successful mathematical activity in school requires at least partial mastery of some of the semiotic systems involved in schooling at the appropriate level. It also involves a number of complex socio-cultural factors that, as is mentioned above, are backgrounded here in order to focus on the mathematical content of sign systems in school. Thus the roles of speaker-teacher, listener-student; the associated power relations; classroom 'contracts'; the aims and purposes of school mathematical activity and tasks, etc., important as they are, are all largely ignored here.

A number of different but interrelated semiotic systems are important in the learning of mathematics. These include, first and foremost, numbers, counting and computation, which together make up the primary semiotic system encountered in school mathematics. This is discussed in detail below. There are several other semiotic systems that students meet in the learning of mathematics over the course of their primary (elementary) and secondary (high) school studies. These include fractions (rational numbers) and their operations, various measures and their means of computation, geometry, probability and statistics, and elementary algebra including the solving of simple equations, both linear and quadratic. Advanced students of mathematics go on to master further, more abstract systems such as calculus or analysis and abstract (axiomatic) group theory, if they continue far enough in their mathematical studies.

There is partial overlap between the different semiotic systems learned in school mathematics; some are learned in parallel and others develop and extend topics met earlier in study. As the discussion of Figure 1 indicated, mathematical topic areas can be, and often are, 'cut up', constituted and defined differently in different contexts and periods of time. Nevertheless, those listed make up a central part of taught mathematics currently straddling the years of study from kindergarten to university.

In the following I concentrate on sketching just the first named semiotic system. It is indicative of some of the most central features of mathematics and its learning, and it forms the basis for virtually all later learned mathematical topics. The intention is to throw some additional light on a well known mathematical topic through the adoption of a semiotic systems perspective.

NUMBERS, COUNTING AND COMPUTATION

This semiotic system initially comprises a particular set of signs, namely numerals, whether they be spoken or written, as well as rules for operating on them and underlying meanings. This is both historically and developmentally the first mathematical semiotic system to emerge, although certain tribal peoples develop ethnogeometry before ethnonumber, e.g., Australian Aboriginal tribes and Navajo Indians (Pinxten, 1987). In order to contrast three important but distinct dimensions of this system I distinguish three different perspectives, the historical, mathematical and developmental perspectives.

Historical perspective on number, counting and computation

The origins of counting and number stretch back many tens or possibly hundreds of thousands of years into prehistory, bathed in the mists of antiquity and mystery. How far back one needs to go to find the origins of speech, language and the full range of communicative activity is unknown. Twenty to fifty thousand years ago repetitive tally-like markings were made by humans in the forms of artefacts (e.g., African and European notched bones) and paintings (e.g., the Lascaux cave paintings). The extent to which these indicate tallies used for calendrical prediction, counting and calculation, ritual markings or cultural symbols used for other unimagined purposes is unknown, but the results of uniform repetitive activity are evident. It therefore appears very likely that some enactive proto-counting activities existed in these pre-historic times.

Linguistic theorists have conjectured, based on comparative analyses of language, that the earliest shared human proto-language of more than 10,000 years ago contained numerical words including 'tik' for one, digit, and finger, and 'pal' for two (Lambek, 1996). The modern ambiguity of the word digit signifying both 'one' and 'finger', links numerals with the embodied action of tallying or the display of a number of fingers. The identification of numbers and counting with body parts and gestures has survived as functional systems of numeration into modern times worldwide among tribal peoples and traditional cultures (Ifrah, 1998; Zaslavsky, 1973).

Prior to written language and numeration it is believed that small clay tokens served as accounting representations of trade goods, with multiple tokens indicating numerical quantities of the goods represented (Schmandt-Besserat, 1978; Radford, 1998). Thus number was represented iconically, through the repetition of tokens or icons, as it is in tallies. The clay tokens were sealed into marked clay 'envelopes', with the pictogrammatic markings indicating the contents. The earliest numeration systems of 5,000 years ago, most notably the Sumerian and Egyptian, represented small numbers by means of tally signs ('I' = 1, 'II' = 2, 'III' = 3, 'IIII' = 4, and so on). Once again, this is an iconic sign form, as the numeral comprises a collection of equivalent parts which has the cardinality of the number denoted, and which therefore can be put into one-one correspondance with any instance of it.

In the transition from the act of tallying to the completed tally marks we also have the shift from action in time to a timeless sign; from verb to adjective or noun; from ordinality to cardinality. Likewise, the iconic marks become part of a fully symbolic sign-system in which more non-iconic numerals are included and the iconic features of the smaller numerals like 'IIII' are backgrounded. In Roman numerals 'IIII' becomes 'IV', and in doing so the numeral for four ceases to bear an iconic relationship to the act of counting to four, or to the cardinal number four, and instead refers within the semiotic system to the result of taking one from five (or in ordinal terms, to the step preceding the fifth step). But of course this is a recent development of less than 2500 years ago. This heralds the next stage of development of abstract numeration systems which occurred in many locations world wide, including Central America, North Africa, the Middle East, the Indian subcontinent, and China. In these locations, different numerals for different denominations were developed leading to a variety of place value systems. Crucial further steps in the development of the modern decimal counting system took place in India and were effectively completed in the 7th century with the introduction and use of zero as a number, circa 600 AD (Ifrah, 1998).

Because of its ubiquity throughout every aspect of modern life the huge magnitude of this conceptual achievement, i.e., the completion of the place value numeration scheme, is easily underestimated. It represents an innovation of the highest level equaled only by (and intertwined with) the development of writing.[2] But despite the widespread myths about its origins, it was not the untrammeled development of human speculative and abstract thought that led to the invention of the semiotic system of numeration and calculation. Numeration arose primarily as an accounting system out of the desire by central rulers of ancient empires to systematically record, document and thus control, wealth, taxation and trade (Høyrup, 1994). In ancient Mesopotamia and Egypt where written arithmetic first originated, it was the knowledge of an elite class of scribes and priests. These practitioners of the arts of early mathematics were primarily the servants of the rich and powerful rulers, serving their purposes. However they also enjoyed refining their arts for their own sake, as well as for educational purposes, as ancient lists of recreational problems demonstrate. In addition, the associations of the study and use of number with religion, astrology and numerology also added mystical and further non-utilitarian dimensions to this knowledge. Wilder (1974) claims that such developments, through positing additional meanings for ciphers, helped to extend numeral signs into fully fledged number signs. Thus numerals as signifiers were firmly welded to individual signifieds within the deeper and more elaborated meaning structures of number and numeration systems that emerged.

In ancient Greece the clear social distinctions between slaves and citizens helped to further, and more fully, bifurcate mathematical knowledge into two types, the secular practical knowledge of calculation and measures and the pure discourse of an elevated leisure class. Logistic, the study and practice of applied numeration and calculation, was viewed disparagingly as a low-level activity suitable only for slaves and lesser beings concerned with day-to-day activities. In contrast, the number mysticism of the Pythagoreans was an abstract and philosophical study entirely divorced from practical calculation. Likewise geometry, despite its utilitarian origins in practical measurement, calculation and applied reasoning, as indicated in the original meaning of its name 'earth-measure', was elevated to the status of a pure science for the exercise of logic and reasoning, for their own sake alone. This gave rise to a more philosophical approach to both geometry and number theory.

Despite their ideological differences, these two dimensions of mathematics, the applied and the pure, have persisted ever since, in a fruitfully synergistic relationship. They have driven developments in both the procedural and the conceptual aspects of number. Practical requirements have driven notational innovations such as the refinement of place value systems and the introduction of negative number notation. Conceptual developments have underpinned these developments, ensuring that the rules of procedure reflect the underlying meaning structures, as well as developing knowledge of other properties. The dichotomy between the pure and the applied dimensions of number has also been bridged by teachers. Like the scribes they systematised knowledge of number for expository purposes, and sometimes pursued pure enquiries for their own satisfaction alone.

As this sketch reveals, the history of number and counting is also the story of the development of semiotic systems of numeration and calculation. In this development the set of primitive signs used became codified and circumscribed, and elaborate and systematic rules for the production of compound signs emerged. The implicit meaning structures underpinning this information technology are those of preserving and extending verifiable operations on sets of tangible objects and on accretions of material produce and substance (e.g., crops, arable land). The operations of numbering (counting) tangible collections of discrete objects or unit measures of continuous material aggregations are predicated on the conservation and replicability of the outcomes. In other words, counting gives rise to invariant and stable semiotic outcomes. Operations of combining and sharing collections of objects form the basis for the numerical operations of addition/multiplication and subtraction/division, respectively.

In acknowledging enactivity, i.e., bodily activity, as a source of meaning for signs I am drawing in part on Bruner's (1960, 1964) synthesis of Piaget and Peirce. Bruner claims that root meanings for signs are often constructed by individuals on the basis of enactive, bodily experiences. Subsequently, Bruner argues, these meanings are further developed through internalisations of iconic representations before being fully represented symbolically. The enactive basis for meaning in mathematics has been more fully developed by Lakoff and Nunez (1997, 2000) who claim that meanings in arithmetic rest on bodily and material metaphors. According to their scheme, numbers are (metaphorically) collections of physical objects, addition is (metaphorically) putting collections of objects together to form larger collections, and more generally arithmetical operations are (metaphorically) acts of forming a collection of objects. In each of these cases it is an imagined agent who performs the (imagined) operations. Enactivism thus provides a useful way of conceptualizing the primitive basis for the meaning structures in semiotic systems And this does not necessarily require subscribing to the epistemological assumptions of cognitivism.[3] Rotman (1987, 1993) has developed a model of mathematical agency within a fully semiotic theorisation that potentially builds on these enactive insights without the assumptions of cognitivism.

Historically, preserving the stability of the original concrete operations motivates the basic and growing meaning structure underpinning general numerical operations. This extends and preserves the stable outcomes of these operations when applied to elaborate and complex compound numerals and repeated symbolic operations on them. The principle of conservative extension, which is first observed in the development of number systems (conserving key principles of the underlying meaning structure while extending the domain of signs or rules in semiotic systems), is a central and enduring dynamic of theory growth throughout all of mathematics, from ancient to modern (Pickering, 1995).

Mathematical perspective on number, counting and computation

Mathematically, the foundational basis of number and counting is reduced to the action of just two primitive arithmetical symbols, ‘0’ for the starting numeral and ‘$'$‘ for the successor symbol. Historically Peano used the two signs ‘1’ and ‘$+1$’, respectively for these primitives (Heijenoort, 1967), but modern scholarship has ruled that there is something unsatisfactory in one primitive sign ($+1$) incorporating the other (1). (Some modern formulations still choose ‘1’ as the starting numeral, as do educational presentations.) By the repeated combination of these two symbols any numeral can be represented and hence defined. Thus by definition $1 = 0'$, $2 = 1'$, $3 = 2'$

and so on. In terms of the primitive symbols 2=$0''$ and 3=$0'''$, and so if the nth successor of 0 is symbolised $0^{(n)}$, then $999 = 0^{(999)}$, and in general, $n = 0^{(n)}$. These definitions seem circular because brevity and manageability is achieved in the metalanguage by using numbers, but it is not a vicious circle. By definition $5 = 0'''''$ (or $4'$), and $0^{(5)}$ is merely a metalinguistic abbreviation. So all of the symbols can be reduced to combinations of the primitives in a non circular and non-self-referential way.

As this shows, at the core of the semiotic system of number and counting is the iterated use of the successor operation. This preserves by analogy the repeated tallying that lies at the heart of number historically (and developmentally). For example, $\text{III} = 3 = 0'''$, i.e., repetitions of the successor operation are denoted analogously to tallies. One outcome of these foundations is that both ordinal and cardinal aspects of number are effectively represented in the semiotic system. For iterated counting (succession) is the basis of ordinal number and hence ordinality, whereas the invariance of the end product of this process is the basis of cardinality. Thus the search for the foundations of arithmetic leads back to the same primitive notions as are found in historical and developmental studies, and has a legitimate place at the heart of the semiotic analysis of number.

Peano arithmetic is a powerful theory in that on the basis of the two primitive symbols and five axioms (together with logical symbols including '=' and axioms of logic and identity), all of the required operations and properties of arithmetic can be established. The axioms of identity are the usual reflexivity ($x = x$), symmetry ($x = y \rightarrow y = x$), and transitivity ($x = y \,\&\, y = z \rightarrow x = z$). Peano's axioms of arithmetic in simplified terms are: 0 is a number ($0 \in N$, the set of Natural Numbers). The successor of any number is also a number ($n \in N \rightarrow n' \in N$). Each number has a different successor ($n' = m' \rightarrow n = m$). Zero (0) is not a successor number ($0 \neq n'$). Last but by no means least, there is the induction axiom (If K is a set, $0 \in K$, and $n \in K \rightarrow n' \in K$, then $N \subseteq K$).[4] The induction axiom, deceptively simple as it is, is the basis for proof by induction throughout all of mathematics,[5] as well as of the inductive definition of functions and properties. This latter property permits all of the usual arithmetical operations to be defined. Thus the binary operation of RHS (right-hand side) addition of a number n to m (i.e., $m + n$) is defined inductively as follows: $m + 0 = m$, $m + n' = (m + n)'$. The binary operation of post-multiplication of m by n is defined by $m \times 0 = 0$, $m \times n' = (m \times n) + m$. (It is readily established that RHS and LHS addition operations are identical, as are pre- and post-multiplication.) The unary operation of exponentiation of m to the power n is defined $m^0 = 1$, $m^{n'} = m^n \times m$.

Foundationally, the very possibility of functions, including these arithmetical operations, is predicated on their well-definedness and the uniqueness of each of their applications and values. Indeed, it is a *sine qua non* in this domain, although such constant and invariant features are far from automatic accomplishments in historical and developmental terms. Not only does induction permit such operations to be defined (well defined even), and their properties to be proven, it also provides an effective means of carrying out the operations. That is it results in a precise algorithm for computing the value for any specific application of these operations. For example, by definition (and in abbreviated form) $2 + 2 = 2 + 1' = (2 + 1)' = (2 + 0')' = (2 + 0)'' = 2'' = 3' = 4$.[6] For such a simple computation this is a somewhat tedious proof, but it demonstrates how every step follows a predetermined algorithm and the number of steps can be predicted in advance from the size of the numbers being operated upon. Thus, $N + M$ can be computed in these basic terms in at most 3 M steps. The regular algorithmic nature of arithmetical operations is a vital constituent in the practicability of arithmetic from its origins in accounting 5,000 years ago to its present near instantaneous use in automated electronic computing, with billions of calculations per second.

The foundational studies of Dedekind, Frege, Peano and others in the late 19th century (Heijenoort, 1967) reduced the number of primitive signs and rules in the semiotic system of number to a minimum. However modern mathematics as taught and practised, both in the academy and in worldly applications, has increased (or rather, maintained that which it inherited from history) the number of primitive numerals from one to 10, namely $1, 2, 3, \ldots, 9, 0$. It has also added a moderately large number (200) of basic facts (concrete rules), where Peano arithmetic has none (only abstract axioms), and utilizes once again Peano's original notation for succession ('+1'). The basic facts assumed are the 100 addition facts: $0 + 0 = 0$, $1 + 1 = 2, 4 + 9 = 13, 9 + 9 = 18$, etc., and the 100 multiplication facts $0 \times 0 = 0, 1 \times 1 = 1, 2 \times 3 = 6, 9 \times 2 = 18, 9 \times 9 = 81$, etc. The utilisation of the increased quantity of primitive numerals in standard notation also depends on the introduction of the place value numeration system, for more than half of these basic facts involve two digit numerals.

The modern academic mathematics of number is couched in philosophically and logically basic terms, as the discussion of foundational work has shown. From this perspective, the complexities introduced by place value notation are regarded as purely notational, and the nature of number is elucidated independently of notational considerations. Notwithstanding this purist view, the origins of which can be traced back to the ancient Greeks, and the philosophers' eschewal of Logistic, much of the historical development of number has concerned the refinement of place-value numeration systems, and the associated algorithms and means of calculation. Evidently a very important dimension of the semiotic system of numerals (and calculation) is the elaboration of the highly economical place value notation system for representing numerals. This offers the means of producing an endless set of compound numeral signs representing numbers of potentially boundless size. For example the compound sign 2134 is a polynomial comprising 2 thousands, 1 hundreds, 3 tens, and 4 units (ones). These numerals of different denominations (themselves the product of a numeral by a power of 10) are combined together by addition. The system of representation of numerals in this general and uniform way has immediate consequences in terms of the relationships between the different positions, such as one Ten is equivalent to ten Ones. More generally, if numeral A is in column n counting from the right (i.e., is a multiple of 10^{n-1}), then it has a positional value of 1/10th of A in position $n + 1$ (i.e., a multiple of 10^n). Thus, a unit in any position has a value 10 times greater (10^n times greater) than one immediately adjacent (removed by n places, respectively) on the right, and 1/10th of the value of a unit adjacent on the left.

A powerful consequence of this notational system, which will not be explored any further here, is that (positive) rational numbers, i.e., the whole or fractional results of the division of one natural number by another (non-zero), can be represented through the utilisation of further positions to the right of the 'Unit column', i.e., utilizing denominations of 10 to negative powers. (Indeed, the generalisation of place value notation as the limit of a power series permits the representation of real and complex numbers.)

The system of compound numeral meanings, as well as that of numerical calculations, is underpinned by a system of rules and principles including the following:

Associativity: $(a + b) + c = a + (b + c) \cdot (a \times b) \times c = a \times (b \times c)$.
Commutativity: $a + b = b + a \cdot a \times b = b \times a$.
Distributivity of $\times$ over $+$: $a \times (b + c) = a \times b + a \times c \cdot (b + c) \times a = b \times a + c \times a$.

From the perspective of Peano arithmetic, these rules are all demonstrable through the use of induction. In the socially widespread semiotic system of arithmetic with compound place-valued numerals, which is utilised throughout practical, educational and mathematical applications, these rules and principles together with those underpinning numerical operations and place value, are part of the underlying meaning structure.

These signs, properties, relationships and principles together with all of the definitions, facts, and rules of the semiotic system support the different algorithms and methods for numerical computation. For example, there are standard taught algorithms such as column addition and multiplication, which are applied to tasks in presentations such as the following.

$$\text{e.g.,}\quad \begin{array}{r} 132 \\ +\underline{792} \end{array} \qquad \begin{array}{r} 263 \\ \times\underline{17} \end{array}$$

Computing the answer to the first of these typically abbreviates the following calculations:

$$\begin{array}{crl}
 & 1 \times 10^2 + & 3 \times 10^1 + 2 \times 10^0 \\
+ & 7 \times 10^2 + & 9 \times 10^1 \pm 2 \times 10^0 \\
\hline
= & (1+7) \times 10^2 + & (3+9) \times 10^1 + 4 \times 10^0 \\
 & (1+7) \times 10^2 + & 12 \times 10^1 + 4 \times 10^0 \\
= & (1+7+1) \times 10^2 + & 2 \times 10^1 + 4 \times 10^0 \\
= & 9 \times 10^2 + & 2 \times 10^1 + 4 \times 10^0
\end{array}$$

The correctness of these algorithms depends on the associativity, commutativity and distributivity of the two operations involved. However to work them correctly only requires knowledge of the 200 number facts mentioned above, as well as knowledge of the algorithmic procedures themselves.

Mathematically, the inverse operations of subtraction and division receive a different treatment from addition and multiplication, despite their analogous original meanings in operations on groups of objects (partitioning and taking away, and repeated sharing or partitioning with removal). This is because mathematically the operations of subtraction and division do not share the 'nice' properties of associativity, commutativity and distributivity that make them so amenable to symbolic manipulation and transformation. Instead, subtraction, of say, 3, must be conceptualised as a compound operation, i.e., the addition of the inverse of 3 ($n - 3 = n +^{-} 3$). By this elegant redefinition subtraction is replaced by the 'nice'. operation of addition at the expense of extending the set N of natural numbers to the integers Z. The integers forms an additive group (under the operation of addition), with the required identity (0), and the properties of closure under addition, inverse operator, and additive associativity.

Developmental perspective on number, counting and computation

The learning of counting and number typically begins as part of language, and persons acquire knowledge and facility of it through participation in language games and conversations situated in forms of life (Ernest, 1998a; Wittgenstein, 1953). Vygotsky (1978) argues that all semiotic functioning is first developed in the young human being through the convergence of several modes of representation, including spoken language, bodily movements associated with drawing and painting, and the use of physical objects as signs, standing for imagined objects in play. Through such modes of expression the power and general properties of the semiotic relation between sign and object, representation and meaning, signifier and signified is first learned and developed.

This can be observed in the development of number competency through the elaboration and convergence of different modes of representation. Number is typically first encountered through repetitive activity in the forms of speech (e.g., 'da', 'da', 'da', . . . ; 'one', 'one', 'one', . . . ; 'one', 'two', 'three', . . .); in repeated bodily movements (e.g., repetitive stepping, pointing, or taking, sweets, pebbles, counters, etc., one by one from a collection); or in making repeated scribbles, brush strokes or tally marks. In such ways counting commonly originates in the child in association with rhythmic and repetitive activity, that is, in enactive representations. As discussed above, this results in basic and deep seated enactive meanings. Initially this gives rise to an ordinal conception of number, as embodied rhythmic activity extended in time, and as iconic sets of marks both resulting from and symbolizing the repetitive activity through which they were created. The experience of completing such activities with their end product, such as a set of marks or a terminal count, also gives rise to a cardinal conception of number as a representation of quantity. Hughes' (1986) experiment with preschool children showed that iconic representations of quantity in the form of tally-like markings can provide an intermediate step between spoken and written (symbolic) signs.

However, it would be a mistake to believe that counting activities and signs emerge in any fixed predetermined order (contra Bruner, 1960, 1964). Enactive, iconic, spoken, written and purely symbolic numerals are utilised and progressively mastered in various overlapping activities and contexts, according to social contingencies as they arise in the child's early experience. Such activities and experiences are primarily communicative ones, in the social space between child and adult and between child and child. What emerges from such communicative activity is partial mastery of the semiotic system of number and counting. This system comprises a particular set of signs, namely numerals, both spoken and written. It is developmentally (i.e., ontogenetically) and historically the first semiotic system in mathematics to emerge. The most important relationship within this system is the basic notion of 'next' in sequence (immediate successor), as identified in foundational analysis. (Subsequently the derived notions of 'further on in the sequence', i.e., formally, 'greater than' and its converse 'less than', also become important). In addition to this internal relationship (within the domain of signs), a further central relationship is the one-one principle in the pairing of signs to objects. These give rise to the basic underlying meaning structure for counting. This is developed and elaborated from participation in social practices of counting, i.e., children seeing others deploy numerals in rule governed ways, and copying or otherwise trying out the use of the signs and rules of the counting system themselves (Gelman and Galistel, 1978; Saxe, 1991).

Just as in history, the acquisition and deployment of a sequence of numeral words or signs is a vital step for the child's development of counting (contrasting with the foundational view that it is secondary). These numeral signs or 'tags' must be deployed in rhythmic iteration with the objects being counted.[7] They must have a stable order so that they embody a single fixed relation of succession. Using them in the enumeration of a set of objects names each object in some order, i.e., gives its ordinal position in the count sequence. When this sequence exhausts the set of objects the last count to be uttered is defined to give the numerosity (cardinality) of the set. This procedure is only useful because of the invariance of the result, a property which must also be learned. The count (cardinality) is conserved by any complete 1-1 count sequence, and neither the order of counting nor the nature or disposition of the objects being counted affects this. Piaget (1952) identified the significance of this accomplishment, and termed it the Conservation of Number. As is well known, in Piaget's (1969) theory this is a key indicator of a child's progression into the stage of Concrete Operations.

In developmental terms, the principles of counting, which provide a basis for rule-governed sign production in context, have been explicitly stated by Gelman and Galistel (1978) as follows.

The principles of counting

1. One-one principle: The child assigns a distinct counting word to each item to be counted.
2. The stable-order principle: The child becomes consistent in her or his use of the counting words or 'tags', even if they have invented their own.
3. The cardinal principle: The child realizes that counting a set of objects results in an end product, i.e., an indicator of the size of the set.
4. The abstraction principle: The child recognises that any set of objects, tangible or imaginary, can be counted, and they do not have to be identical.
5. The order-irrelevance principle: objects can be counted in any order, and the result will still be the same.

In elementary mathematics, counting activities will often at first concern manipulating and working with concrete, graphical and symbolic representations of number. There will often be ordering by size, matching (e.g., sets of objects), counting (e.g., repeated motifs in diagrams), and numeral recording activities set for the learner.

Learning the semiotic system of number and counting means that a learner is able to participate in certain social activities and practices and within them can utter or produce number signs appropriately. Being able to participate in such counting conversations, activities and language games means that the learner can use the semiotic system creatively. Indeed, the mastery of any self-sufficient part of a semiotic system, with its signs, rules and meaning structure, enables its user to employ the signs creatively. Every act of counting requires decisions and choices such as where to start the count and how to sequence a set of objects in the environment or imagination so as to respect the principles and meanings involved. Thus counting, as routine and automatic as it can become in familiar situations, begins as a creative act of problem solving.

The standard teaching sequence for the semiotic system of number typically follows on from the partial mastery of counting with the introduction of number operations. (Inevitably, counting capabilities remain incomplete until after the mastery of the rules of production of compound numerals, based on place-value notation with its implicit numerical operations). The first such operation is addition, beginning with the combined counting of two sets of objects. Thus to compute $3 + 8$ (or rather to give a total count for sets of 3 and 8 objects) a child typically counts the first set (1, 2, 3) and then continues the count for the second set (4, 5, 6, 7, 8, 9, 10, 11), thus treating the two as a single partitioned set. At this stage small two digit numbers such as 10 and 11 typically have a pre-place value meaning given by their position in the counting sequence, rather than through their composition as sums of tens and units, i.e., their place value meaning.

The primordial meaning of addition is that of the combined count of two sets of objects (or one partitioned set). This is a natural extension of the principles of counting, as indicated above, which also guarantee a stable answer to this operation. In the subsequent development of the semiotic systems of school mathematics this base meaning is added to and deepened in terms of its application to new and extended sets of signs, rules and underlying meaning structures. In general, as was discussed previously, such changes and developments in rules and meanings are a source of problems in the learning of mathematics. But they are also an inescapable source of the cognitive and expressive power of the semiotic systems of mathematics. The shifts and growth of meaning structure and rules evolve in parallel and exemplify the role of metaphor in extending meanings in language.

Some of the subsequent development of competence in addition may be spontaneous. As researchers have discovered, it is not only in the basic operations of counting that children's procedures are creative. As children learn the elementary number operations of addition (and later subtraction, etc.) on small numbers they also typically invent curtailed procedures. Carpenter and Moser (1982), Fuson (1982), Ginsberg (1977), and others have identified a multi-stage progression which children use when using counting in order to add two numbers, say $N + M$. First, there is the 'count all' strategy, discussed above, in which children start by counting out N and then follow it with M further counts (providing a sequence of $N + M$ numeral utterances or counts in total). The child may first have been shown (and practised) performing addition by counting all of two sets of physical tokens (enactive representation of task), counting all of two sets of drawn tokens (iconic representation, possibly worked enactively), and then counting all of two numbers (symbolic representation of task). Even in this last case there is a vestigial enactive element in counting, uttering or thinking a sequence of numerals, often indicated by minute bodily movements, such as finger, lip, or eye movements.

The second stage is 'counting-on', where a child starts with the count of N (without recounting it) and follows this with M further counts (giving a sequence of $1 + M$ numeral utterances in total, e.g., for $3 + 8$ counting 3, 4, 5, 6, 7, 8, 9, 10, 11). The child may be taught or shown that counting on shortens the procedure and is a more efficient way of completing the task, or may infer it during the experience of an extended number of addition tasks. In reaching this stage the child has thus spontaneously realized or learned that the same end result can be achieved by an abbreviated and more economical procedure. Where spontaneous, this is creativity at work.

In the third stage, 'counting-on from larger', if $M > N$ then the child commutes $N + M$ to $M + N$, and performs the transformed calculation by 'counting-on' as before. This involves $1 + N$ utterances plus the initial, possibly unspoken, commutation, e.g., for $3 + 8$ counting 8, 9, 10, 11. Again this can be the result of creative problem solving where the child combines known procedures to make a new one, having learned from multiple experiences that larger number first is less effort. Thus the child has been taught, shown or self-induced the commutativity of addition.

In a fourth stage the child derives the answer to $N + M$ from known facts. At this stage the child has mastered some but not all of the 100 one digit addition facts. Chronometric analysis indicates that children memorise certain additive number facts first, for example $2 + 2$, $3 + 3$, $4 + 4$, ... (Resnick and Ford, 1981). Thus, children often come to know 'doubles' such as N + N for small values of N before most other number facts, and they use these to derive other facts. Thus they might compute $4 + 5$ as $4 + 4 + 1 = 8 + 1 = 9$. More generally, if $M = N + K$ they might choose to compute $N + M$ as $(N + N) + K$ (requiring $1 + M - N$ utterances).

However, in the case of the example $3 + 8$, a learner might instead transform this into $3+[7+1]$, $[3+7]+1$, $10+1$, 11, using the 'make ten' bridging strategy (Thompson, 1998, 1999) since using the double $3 + 3$ would be inefficient, not curtailing the calculation sequence. Deployment of this strategy implies a grasp, probably operational and implicit, of the associativity of addition. Each such derivation from known facts is a creative act of problem solving, at least initially. It requires knowledge of the partition of 10 facts (e.g., $1 + 9 = 2 + 8 = 3 + 7 = \cdots = 10$). It involves selecting one of these known facts anticipated to be helpful as a starting point for the given sum in the light of available strategies and then deriving a sequence of transformations to link them. It also necessitates a degree of metacognitive monitoring and self-regulation to make an efficient strategic choice. As before, such a use of facts to derive others may be a self induced strategy from extensive experience in additive tasks in school, or may be taught or modeled by more knowledgeable others, or a combination of the two.

In the fifth stage the child knows all of the 100 addition facts and retrieves $N + M$ directly from memory in a single rapid mental operation. Transition to this final stage typically takes several years of practice with the semiotic system of number, applied to teacher provided addition tasks and other incidental calculations, and is attained later or never by some 'learning disabled', but otherwise normal, students. However even when this final stage is reached with respect to the 100 addition facts, the methods and strategies used in the earlier stages continue to be employed creatively in mental operations. The flexibility and choice of strategies involved are those that have been identified as the general strategies of problem solving, that is, procedures that guide the choice of what knowledge and skills to use at each stage in problem solving. As a number of authors have noted, e.g., Fischbein (1994) and Plunkett (1979), such mental methods typically involve repeated operations to facilitate the application of relatively simple operations. Typically in such working, composite (multi-digit) numerals are decomposed and recomposed creatively using a variety of equivalence transformations including the number facts and rules of associativity, commutativity and the distributivity of operations identified above, as best suits the particular calculation (Plunkett, 1979; Thompson, 1999). Computations may involve single or multiply branching sequences of multiple decompositions, distributions, partial calculations and recombinations. The process is an often fleeting sequence of partial computations represented by mental imagery or written signs directed towards the desired result.

The learner masters the above sequence of addition skills, usually in the listed order, in response to demonstrations by, or shared tasks with, more capable others (within the learner's Zone of Proximal Development, after

Vygotsky, 1978). In this progression there are many experiences that contribute to the learner's developing mastery over the symbolic addition tasks. These include work with manipulatives and iconically presented tasks, as well as oral work and a wide range of symbolic tasks solved mentally, with intermediate recording or with the electronic calculator. The symbolic tasks can include written addition, subtraction (as well as multiplication and division) tasks in which numerals are displayed horizontally or vertically, in column form. The meanings that children attach to the operations of addition and subtraction are intimately related, and so work with tasks utilizing both of the operations contributes to the development of their addition skills. The tasks typically also include word problems, which the child must analyze and translate into an arithmetical task before operating on any numerals. Children will have most of these experiences in the classroom, but other significant areas of activity occur in the home and other out-of-school locations. Likewise the more knowledgeable guiding other will most often be the elementary school teacher and sometimes a classroom peer, but may also include parents and others in the out-of-school locations.

This account indicates just some of the rich complexity underlying an individual's mastery of the semiotic system of number. Being able to add small numbers, as described, and ultimately, being able to use these addition facts for other tasks is the culmination of a process which extends over several years. In this process the learner is internalizing some of the central functions and structures of the number system, i.e., building up the underlying meaning structure. This internalisation necessitates the learner to be continually engaging in conversation, making public utterances and performances, deriving feedback from others, incorporating confirmations and corrections in his or her performance and functioning, which helps to shape the child's emerging powers. Although the focus in this account has been on the mastery of the semiotic system of number with the social dimension backgrounded, the learner is also learning to read, understand and respond to the social contexts of number. Thus the learner as an apprentice mathematical subject also develops the following abilities.

First, there is the ability to identify the particular arithmetical tasks or activities as set or imposed by others, including in the classroom and at home. Such identification may also be needed where the tasks are embedded in self-subsistent social forms of life, such as giving the correct coins in shopping. This capacity involves 'reading' (interpreting) the context, that is being aware when a task is signaled within a social situation. The identification function may become unconscious as the child learns to recognise, understand and engage with the task in an overall undifferentiated act within a familiar context. However some of the different situated practices involved, e.g., performing calculations in the classroom and calculating correct change in shopping, may be learned as separate practices and may not lead to transfer of learning or competencies, as both cognitive-based research (Nunes et al., 1988; Nunes, 1992) and work in situated cognition (Evans, 2000; Kirshner and Whitson, 1997; Lave and Wenger, 1991) indicate.

Second, there is the child's acquired ability to accept the task imposed by another, or necessitated by the situation, and to engage with its goal directed nature. This involves being able to accept and internalise the goal of a task and behave as if it were the subject's own personal goal. This capacity does not exclude the possibility of task rejection or goal refusal by the learner, and this possibility always remains open. However, acceptance can take place without the child consciously making a decision, or indeed having any awareness of tacit compliance through subserving his or her will to that of a directing other. Acceptance and compliance with others' will is something children learn from the earliest stages from their reliance on primary caregivers. In my view the existence of the phenomenon of hypnotism, with its directorial and suggestive potency, suggests that this a deeply entrenched human capacity. Power is ever-present in conversation and interpersonal communicative activity and roles, although power in this sense is productive, and only problematic when used abusively (Foucault, 1981, 1982).

Third, learners also develop the ability to select from their personal repertoire of knowledge, skills and procedures to perform appropriate functions in response to task representations. Such functions often involve the transformation of texts, the production of oral responses or other means, in order to attain the goal of the task. Responses utilise a variety of modes of representation including literal, symbolic or iconic inscriptions, perhaps also enactions, gestures, utterances and their combinations. The capacity to perform an appropriate semiotic function does not preclude the possibility of failure in selecting an appropriate function, in performing an appropriate function correctly, or in attaining the task-goal, in any particular instance. Any performance is creative, at least initially, with the concomitant risks and satisfactions that the deployment of skill involves.

Successful appropriation and deployment of the semiotic system of numerals and calculation is a process that takes several years and involves the growing mastery of a range of signs, meanings and rules of calculation which are combined in different computations in creative ways which can become routinised with practice. This encompasses the four main algorithms (the 'four arithmetic operations') and all the modes and contexts of presentation outlined above, and constitutes the central emphasis in elementary school teaching of mathematics.

These, then, are some of the powers and capacities that the student appropriates and develops in attaining mastery of semiotic system of number. As this account shows, they are inescapably bound up with the interpretation and production of mathematical text. In the context of schooling, such texts become increasingly formalised over the passing of the years. In addition to gradual mastery of the semiotic system of number, a growing mastery of the broader and more general rhetoric of school mathematics is also a major expectation of school mathematics (Ernest, 1998a,b; 1999). This is not to exclude the possibility that each semiotic system of school mathematics has its own specific modes of formal representation, presentation and rhetorical demands.

One further aspect of the semiotic system of number from a developmental perspective needs to be clarified. This account has focussed on the counting and calculational aspect of number work in children's learning. In the history of mathematics, in addition to the practical, applied-driven and calculational aspect of number there is also pure number theory, pursued initially for its own challenges and aesthetic reasons. School arithmetic also has elements of this dimension, which is a further part of the semiotic system of number. While children are developing counting, numeration and calculational skills they are also typically introduced to number properties and patterns. Children learn about even and odd numbers, primes and composites, and sometimes, later, square, triangular, and even perfect and abundant numbers. Further *ad hoc* properties may be introduced in number investigations, such as Happy and Sad numbers (Jeffrey, 1981). Even and odd numbers are represented iconically through rectangular arrays of dots (just as the Pythagoreans did). Even numbers give two even lines of dots, whereas odd numbers always have one extra dot. Composite numbers give rectangular arrays (square numbers make square arrays), and primes can only be drawn as lines. The introduction of these concepts and their properties helps to children to develop the relational properties of number introduced by the operations of addition and multiplication. These properties extend the meaning structure underpinning numerals and number representations, as well as the nexus of meanings underpinning the rules of the semiotic system.

These additional aspects are part of the semiotic system (or systems) of number, which utilises a variety of signs and representations including iconic ones (as here) as well as enactive, verbal and symbolic ones, discussed above. This raises a deep question. These aspects of number, presented here and above with greater and lesser thoroughness, are they all part of the same semiotic system of number? Developmentally, is there only one semiotic system of number that children, to a varying extents, master? More generally, are the different accounts of the semiotic systems of number from historical, foundational and developmental perspectives accounts of a single system, or are there many systems involved? My answer is that there is not a single, uniquely defined semiotic system of number, but rather a family of overlapping, intertransforming representations of constituting the semiotic systems of number. It would make no sense to claim the existence of a single, fixed and essential representation, for in the developmental arena alone there are multiple representations utilised by teachers at different stages in teaching, as well as the varying systems mastered by children themselves. Some aspects of this complexity were indicated in the discussion around Figure 1, above. Likewise, from a historical perspective, a multiplicity of systems can be described, and it cannot be said by any means that they converge into a single systematic representation. The same is true foundationally. Similarities between different system representations abound but each version although a representative of the family of overlapping, intertransforming representations of semiotic systems of number is also a contingent function and product of its social context and purposes. Unity and diversity co-exist simultaneously.

CONCLUSION

This paper has analyzed elementary arithmetic from a semiotic systems perspective. It has shown the importance and interdependence of each of the components of signs, rules and meanings whether viewed historically, foundationally or developmentally. Despite the obvious differences between these perspectives, there is considerable convergence between them. This is not surprising in view of the central role of the semiotic system of number in each of these domains.

Following Saussure, diachronic and synchronic modes of analysis can be applied to semiotic systems. The diachronic mode is a long term view focussing on the development of the system itself, chronologically. Each of the three perspectives presented has emphasised this view. The historical perspective treated key stages in the emergence, social function and development of systems of number, counting and calculation over five millennia. The foundational perspective provides a logical narrative of the emergence and development of number concepts and functions beginning with primitive notions and then defining new numbers and operations and establishing their properties by means of deductive proofs. This perspective is positioned as timeless logico-philosophical narrative, but it too has its own historical development with different mathematicians providing different and increasingly explicit accounts of the foundations of number. An extended historical account of foundations could be provided including the complete systems or contributions of Pythagoras, Euclid, Al-Khwarizmi, Pascal, Fermat, Dedekind, Peano, Frege, Russell, Brouwer, Hilbert, Gödel and others (Benacerraf and Putnam, 1964; Heijenoort, 1967; Grattan-Guiness, 1994). Such an account would forground the foundational changes in the evolution and development of semiotic systems of number over nearly three millennia. Last but not least, the developmental perspective focuses on children's learning and mathematical functioning, and is a most dramatic picture of rapid development as children grow from babies with no speech, let alone arithmetic, to fully fledged calculators and junior number theorists by 10 years of age.

The synchronic view focuses on the role and function of the semiotic system of number in live conversation. Deployment of this system in each domain involves the two primary social roles within conversation, sign receiver (listener or reader), and sign producer (speaker or writer). In each case, sign construction or utterance involves the production of a sequence. Historical records provide evidence of completed calculations often concealing the sequential nature of their derivation. But from the methods embodied in the range of completed calculations discovered, it is clear that comparable sequential processes in calculation and the production of text are present throughout history. The process of counting requires the utilisation of a sequence of signs (numerals). Numerical computation and calculation requires the production of an elaborated sequence of signs, ultimately arriving, if successful, at a terminal text 'the answer'. Sometimes this sequence consists of the elaboration of a single compound sign (e.g., carrying out a 3 digit column addition). This means that a compound sign involving symbols becomes also an icon in which annotations, crossings-out and the spatial dispositions of constituents signify part of the composite meaning. Sometimes a computation involves a sequence of distinct, spatially extended or temporarily distinct signs such as in mental calculations. In each case, there is a given text, either chosen or imposed, specifying the numerical task. This text may include signs from the semiotic system to be transformed as the starting point for the task, e.g., two numerals to be added, a word problem incorporating numerals to be solved, or in a foundational example, a proof of a property such as a proof by mathematical induction that $\times$ is commutative over N. It may draw upon everyday words as synonyms for signs in the semiotic system, such as number words in word problems. In some tasks the text does not include basic signs from the semiotic system, e.g., 'count the animals in the picture', or has indications beyond the text, e.g., 'count the children in your class'. In response children must introduce the numeral signs themselves, although in these examples the 'count' instruction is a rule governed sign denoting an operation within the semiotic system of number.

In performing the task a sequence of signs is produced in conformity with the rules of the semiotic system. In counting, these are the rules of counting, and these rest on a meaning structure that conserves the cardinality of counts as invariant. In calculating, these are the rules associated with arithmetical operations (commutativity of $+$, etc.). Calculations are typically presented as sequences of numerical terms, each term derived from predecessors by the rules of the system. The meaning structure underpinning the rules of calculation concerns the preservation of numerical value. Multidigit column subtraction algorithms provide a good illustration, although in this case the sequence tends to be produced as successive annotations over time, resulting in a single over-inscribed composite. These algorithms are based on individual subtraction operations applied within 'columns', i.e., treating each denomination of powers of ten separately. Both Equal Addition and Decomposition involve compensatory adjustments within compound numerals in adjacent place value positions so that the overall operational outcome is preserved. The Equal Addition algorithm transforms the task into a different but equivalent subtraction, through a translation 'up the number line' to a different 'sum' (task) with the same fixed number difference. In contrast the Decomposition algorithm does not change the subtraction. In each case, the value of the numerical difference is preserved invariant.

In foundational treatments, the production of deductive proof sequences plays a central role. These consist of sequences of sentences, each derived from predecessors by the rules of the system. The meaning structure underpinning the rules of proof concern the preservation of the truth value of sentences in each deduction, and hence along the length of the proof sequence. This may seem very different from the calculation examples discussed above, but there is a strong analogy between calculation and proof sequences that is under-remarked in the literature. Calculations utilise the term as a basic unit of meaning (and which is transformed), whereas deductive proofs use the sentence as a basic unit. However, there are equivalence transformations between calculations and proofs. A calculation sequence of the form $s, t, u, v, \ldots,$ can be represented as a deductive proof of the form $s = t, t = u, u = v, \ldots,$ in which each identity asserts that numerical values of adjacent terms are preserved identically in the calculation. Likewise, a deductive proof of the form $P, Q, R, S, \ldots,$ can be represented as a series of terms, i.e., the values of the truth value function f defined on numerical representations of true and false sentences to give the values 1 and 0, respectively. For a valid proof these values must be $f(P \to Q) = f(Q \to R) = f(R \to S) = \cdots = 1$. The formal details are messy and omitted here (see Gödel, 1931 for the introduction of arithmetisation of logic, and Kleene, 1952) but the principle is both simple and sound.

I have suggested that mathematical activity understood semiotically involves the comprehension and production of mathematical signs with the following properties:

1. The basic signs and symbols are drawn from a limited 'alphabet' but are combined to make a large number compound signs.
2. The production of signs (always in conversation, be it 'live', imagined, or otherwise attenuated) involves the production of sequences, including linear, multi-dimensional and juxtaposed or over-inscribed sequences of signs.
3. Rules and constraints limit the introduction of signs at each stage in such sequences (i.e., determine which are legitimate and are accepted as such in the conversation).
4. Despite these constraints, which can be much tighter in formal mathematical semiotic systems than in spoken and written language use, there is always an element of creativity in producing such sequences.
5. Supporting the sign use is a meaning structure that gives meaning to the signifiers and underpins the rules of sign production in terms of preserving of key aspects of meaning.

These claims have been directed at a specific semiotic system, namely that of number and calculation. However, the extension of these properties, and semiotic systems in mathematics in general, is part of an ongoing project to develop a semiotics of mathematics and mathematics education (see Ernest, 1994a, 1997, 1998a, 1999, 2003, in-press). In the present context they have served to demonstrate how insights from the disparate historical and foundational dimensions of number highlight features of the signs, rules and meaning structures of arithmetic, adding to understanding of the developmental study of number for the teaching and learning of mathematics.

NOTES

1. Here and below I make repeated references to 'sign'. I have in mind primarily the de Saussurian concept of sign, comprising both signifier and signified. However nothing I say presupposes this as opposed to the Peircean three part definition of sign. What I do presuppose is shared by both theoretical perspectives. Namely that a sign involves an inscription, expression or other means of signification (signifier), and refers to some content or meaning (signified). However this is susceptible to multiple interpretations (similar to Peirce's concept of the interpretant).
2. Rotman (1987) has argued that not only was the introduction of Zero a giant semiotic step for humankind in its metalinguistic completion of the place value system, but that it also closely parallels, in conceptual terms, the introduction of paper money in finance and

the vanishing point in artistic perspective. These audacious analogies are surprisingly convincing in Rotman's argument, despite their superficial implausibility.

3. Cognitivism resembles the psychological approaches rejected at the outset of this paper for foregrounding and privileging the mental structures of individuals. From the perspective of semiotics, cognitivism and enactivism helpfully identify embodied personal experience as a key contributor to meaning-making, but they neglect the importance of social communicative activity in the formation both of individuals themselves and in their construction of meaning. The semiotics-based position adopted here sees sign based activity as straddling the individual-social and private-public boundaries and contributing to the formation of both sides of these dichotomies.
4. For simplicity of expression, universal quantification is assumed for free variables *n, m, x, y, z*.
5. In any field of mathematics over any domain of numbers or other objects, any well defined property or expression (p(n), say) that has one natural number variable, no matter how many other variables of whatever type, is eligible for proof by induction. For the induction axiom can be applied to $K = \{n \in N : p(n)\}$ in a standard proof by induction.
6. It is interesting to note that underlying the proof procedure exemplified here is a formal version of the 'counting on' procedure for addition discussed in the next section.
7. There is a different visual number recognition technique which is instantaneous rather than a temporal procedure, namely 'subitising'. This only applies to small numbers of objects in a recognisable geometric configuration, such as 4 dots in a square pattern. It usually develops after the acquisition of basic counting skills, but it can be taught independently of the rules of number, even to some animals.

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