RAYMOND DUVAL

A COGNITIVE ANALYSIS OF PROBLEMS OF COMPREHENSION IN A LEARNING OF MATHEMATICS

ABSTRACT. To understand the difficulties that many students have with comprehension of mathematics, we must determine the cognitive functioning underlying the diversity of mathematical processes. What are the cognitive systems that are required to give access to mathematical objects? Are these systems common to all processes of knowledge or, on the contrary, some of them are specific to mathematical activity? Starting from the paramount importance of semiotic representation for any mathematical activity, we put forward a classification of the various registers of semiotic representations that are mobilized in mathematical processes. Thus, we can reveal two types of transformation of semiotic representations: treatment and conversion. These two types correspond to quite different cognitive processes. They are two separate sources of incomprehension in the learning of mathematics. If treatment is the more important from a mathematical point of view, conversion is basically the deciding factor for learning. Supporting empirical data, at any level of curriculum and for any area of mathematics, can be widely and methodologically gathered: some empirical evidence is presented in this paper.[1]

KEY WORDS: cognitive paradox, figural organization, knowledge object, language, mathematics learning, recognition, multifunctional and monofunctional registers, non-congruence, representation, representation conversion, semiotic representation, semiotic system, thinking processes, treatment

How can we understand the difficulties, frequently insurmountable, that many students have with comprehension of mathematics? What is the nature of these difficulties? Where are they located? These questions have taken on a particular magnitude and importance with the recent pressure for more initial mathematical training to be given to all students in order to prepare them to face a technological and computer-oriented environment of perpetually increasing complexity. They are both an educational challenge in classrooms and a theoretical challenge to research on the development and learning of mathematical knowledge. The processes of mathematical knowledge acquisition are so complex that quite different approaches seem required. The most predominant, and sometimes opposite, are the epistemological and the educational. But they have in common the use of the notion of representation to characterize the kind of phenomena that occur in any knowledge process or that constitute it.

This basic notion of representation is very old and accurate. A representation is something that stands for something else. But at the same time this notion can be elusive or too formal. What is the nature of this "something

Educational Studies in Mathematics (2006) 61: 103–131
DOI: 10.1007/s10649-006-0400-z

standing for . . .”? You can get quite a wide range of answers, depending on whether you consider the representations with regard to the concrete individual and his or her experiences, to the mind structures, or on the contrary, to the knowledge objects with their specific epistemological requirements (Hitt, 2002). Thus, representations can be individuals’ beliefs, conceptions or misconceptions to which one gets access through the individuals’ verbal or schematic productions. This answer, first developed in two major studies of Piaget (1923, 1926), is now one of the major methodological and theoretical frameworks for investigating and explaining mathematical knowledge acquisition. But representations can also be signs and their complex associations, which are produced according to rules and which allow the description of a system, a process, a set of phenomena. There the semiotic representations, including any language, appear as common tools for producing new knowledge and not only for communicating any particular mental representation. This answer, which has been progressively developed since Frege and Hilbert with regard to epistemological and metamathematical requirements, has also taken on a great importance in the investigation about cognition (Duval, 1998a). Any research about the learning of mathematics involves some theoretical choice about the possible relationship and the respective role of these quite opposite kinds of representation, which are all “standing for something else”, that is represented objects of knowledge.

It seems obvious that research about the learning of mathematics and its difficulties must be based on what students do really by themselves, on their productions, on their voices. But how can we analyze the processes of knowledge acquisition from the students’ conceptions and find out the sources of their difficulties? Representations are only the surface results of the functioning of deep mind structures that do not depend on the actual awareness of individuals (Piaget, 1967, pp. 78–79). Underlying the two quite opposite kinds of representation, there is an organization of cognitive structures that make individuals able to perform the various kinds of knowledge activity (Duval, 1996a). Thus, the characteristic feature of a cognitive approach is to seek first to determine the cognitive functioning underlying the various mathematical processes. In order to determine the origin of the students’ incomprehension we must first determine the cognitive conditions that make comprehension possible. For that we must ask the question:

1. What cognitive systems are required and mobilized to give access to mathematical objects and at the same time make it possible to carry out the multiple transformations that constitute mathematical processes?

It is generally assumed that the way of thinking is basically the same in the different areas of knowledge even though mathematical knowledge is more abstract, and even if specific language or coding are used in mathematics. Observations I have practiced in classrooms and outside the classroom for many years led me not only to change from an approach focusing on students' conceptions (Duval, 1983) to a cognitive approach, but also above all to ask the question:

2. Is the way of thinking the same in mathematics as in the other areas of knowledge? In other words, does mathematical activity require only the common cognitive processes or, indeed, certain very specific cognitive structures whose development must be taken care of in teaching?

This issue about the learning of mathematics has a great significance if the goal of teaching mathematics, at the primary and secondary level, is neither to train future mathematicians nor to give students tools, which can only possibly be useful to them many years later, but rather to contribute to the general development of their capacities of reasoning, analysis and visualization. In any case, it makes it necessary to consider semiotic representations at the level of mind's structure and not only with regard to the epistemological requirement for getting access to knowledge objects (Duval, 1995a, pp. 3–8, 15–35). And from this cognitive approach it appears that the opposition between mental representations and semiotic representations is no longer relevant, because it rests on the confusion between the phenomenological mode of production and the kind of system mobilized for producing any representation (Duval, 2000a, pp. 59–60).

I shall present here some of the main results I have obtained. They are related on the one hand to the predominant role played by transformations of semiotic representations in any mathematical activity, and, on the other hand, to the kind of the semiotic system used for these transformations. The cognitive complexity underlying the thinking processes in mathematics lies in the fact that there are two quite different forms of transformations that are never taken explicitly into account in teaching. And from the mathematical point of view, one of them commands the most attention, while it is the other that causes the greatest difficulties for students. After a description of the various cognitive processes required by mathematical thinking, I will present some empirical data for showing how these two kinds of transformations are specific and independent sources of incomprehension in the learning of mathematics.

1. WHAT CHARACTERIZES MATHEMATICAL ACTIVITY FROM A COGNITIVE POINT OF VIEW?

When trying to analyze what constitutes mathematical comprehension and to explain the obstacles to understanding that students experience, people often bring up the concepts and their epistemological complexity. And this epistemological complexity can be explained by the history of their discovery. But such an approach is not sufficient to characterize what is novel and specific to thought processes in mathematics in contradistinction to other domains of scientific knowledge such as astronomy, biology, etc. The difference between the cognitive activity required for mathematics and that required for other domains of knowledge is not to be found in the concepts – because there is no domain of knowledge that does not develop a set of more or less complex concepts – but in the following three characteristics.

1.1. *The paramount importance of semiotic representations*

One has only to look at the history of the development of mathematics to see that the development of semiotic representations was an essential condition for the development of mathematical thought. For a start, there is the fact that the possibility of mathematical treatment, for example calculation, depends on the representation system. Because the leading role of signs is not to stand for mathematical objects, but to *provide the capacity of substituting some signs for others*. Thus, there is an enormous gap between these two kinds of number representation: stick or stroke collections and base systems within which the position gives the meaning. And here the trouble appears with *this very strange sign "0" which does not belong to the chosen base, but to a powerful semiotic system of number representation.* Thus, because a true use of decimal notation system is not necessary for working with small integers and for doing additive operations, we can consider that the decimal notation "*10*" stands for the quasi-material representation "IIIIIIIIII" of the number "*ten*" and gives the meaning. But beyond that, does not its use require understanding the way in which the used representation system functions? For example, in such expressions as 38.45×10; 38.45×100 or $38.45 : 0.1$; $38.5 : 0.01$? How many young students do truly get to this stage of comprehension? And students' acquisition of these systems is not simple. One might think that employing the number system from the beginning of pre-school would make its use progressively more transparent. French national assessment surveys (MEN, 1993, 1997) showed that that is not yet the case at the beginning of secondary school: only one student in three appeared to have grasped the functioning of the decimal system and to be able really to make use of its possibilities in order to succeed with a set of items about the simplest operations of multiplication and division of decimals ($38.45 \times 10 : 45 \times 0.1$). In addition, there is the fact that mathematical objects, starting with numbers, are not objects that can be directly perceived or observed with instruments. Access to numbers is bound to the use of a system of representations that allows them to be designated.

But the key point is not there. The part played by signs, or more exactly by semiotic systems of representation, is not only to designate mathematical objects or to communicate but also to work on mathematical objects and with them. No kind of mathematical processing can be performed without using a semiotic system of representation, because mathematical processing always involves *substituting some semiotic representation for another*. The part that signs play in mathematics is not to be substituted for objects but for other signs! What matters is not representations but their transformation. Unlike the other areas of scientific knowledge, signs and semiotic representation transformation are at the heart of mathematical activity. Why?

1.2. *The cognitive paradox of access to knowledge objects*

From an epistemological point of view there is a basic difference between mathematics and the other domains of scientific knowledge. Mathematical objects,[2] in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments (microscopes, telescopes, measurement apparatus). The only way to have access to them and deal with them is using signs and semiotic representations. That means that we have here only a single access to the knowledge objects and not a double access, mainly non-semiotic and secondarily semiotic, as is the case in the other areas. This very specific epistemological situation of mathematics changes radically the cognitive use of signs. Any learner is faced with two quite opposite requirements for getting into mathematical thinking:

– In order to do any mathematical activity, semiotic representations must necessarily be used even if there is the choice of the kind of semiotic representation.
– But the mathematical objects must never be confused with the semiotic representations that are used.

The crucial problem of mathematics comprehension for learners, at each stage of the curriculum, arises from the cognitive conflict between these two opposite requirements: *how can they distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representations*? And that manifests itself in the fact that the ability to change from one representation system to another is very often the critical threshold for progress in learning and for problem solving.

1.3. *The large variety of semiotic representations used in mathematics*

Highlighting the utmost role of semiotic representations in mathematical activity, which necessarily involves sign substitution, is not sufficient. Mathematical activity needs to have different semiotic representation systems that can be freely used according to the task to be carried out, or according to the question that is asked. Some processes are easier in one semiotic system than in another one, or even can be made in only one system. But in many cases it is not only one representation system that is implicitly or explicitly used but at least two. Thus, in geometry it is necessary to combine the use of at least two representation systems, one for verbal expression of properties or for numerical expression of magnitude and the other for visualization. What is called a "geometrical figure" always associates both discursive and visual representations, even if only one of them can be explicitly highlighted according to the mathematical activity that is required. Then, students are expected to go to and fro between the kind of representation that is explicitly put forward and the other that is left in the background of this discursive/visual association that forms any geometrical figure. And this association is cognitively complex because in most cases it goes against the common association between words and shapes and because its use runs against the perceptual obviousness (Duval, 1998b, pp. 38–44).

Mathematics is the domain within which we find the largest range of semiotic representation systems, both those common to any kind of thinking such as natural language and those specific to mathematics such as algebraic and formal notations. And that emphasizes the crucial problem of mathematics comprehension for learners. If for any mathematical object we can use quite different kinds of semiotic representation, how can learners recognize the same represented object through semiotic representations that are produced within different representation systems? More deeply than the epistemological difficulties peculiar to each introduction of new concepts, would not the most recurrent obstacles in mathematics comprehension come from these specific ways of thinking involved in any mathematical activity?

2. HOW TO ANALYZE THE THINKING PROCESSES INVOLVED IN MATHEMATICAL ACTIVITY?

The role of semiotic representations is not confined to designating objects, to standing for something else, or to being themselves considered as objects. Their use is determined by the possibility of mathematical processing that they permit. Whatever semiotic representations are used, they can be changed into other semiotic representations without the support of new data or empirical observations. Otherwise, the basic cognitive operation of substituting some semiotic representation for another would not be possible. But that depends on the semiotic system within which semiotic representations are produced. Each semiotic system provides quite specific possibilities. The variation of "capacity", which was mentioned by Peirce (CP: 2.228) for the *representamen*, is not on the level of particular representations, but on the level of the semiotic system within which they are produced. Thus, for analyzing the complex and specific thinking processes that underlie mathematical activity, we must take into account the differences between the various semiotic representation systems that are used. Do these differences play an important part in the mathematical processes? Whenever we analyze the student's difficulties and blocks in learning of mathematics, we are faced with this issue.

2.1. *How to describe the various mathematical processes?*

Given the cognitive paradox of access to knowledge objects in mathematics, such a description must be supported by the variety of semiotic representation systems that are used and by the specific "capacity" of each one for performing mathematical processes.

The most widespread way to classify is to oppose language, natural or symbolic, and image. However, this is general and above all it is far from sufficient. There is also another essential difference that is very often missed. Some semiotic systems can be used for only one cognitive function: mathematical processing. On the other hand, other semiotic systems can fulfill a large range of cognitive functions: communication, information processing, awareness, imagination, etc. (Duval, 1995b, pp. 89–90). This functional difference between the various semiotic representation systems used in mathematics is essential because it is intrinsically connected with the way mathematical processes run: within a monofunctional semiotic system most processes take the form of algorithms, while within a multifunctional semiotic system the processes can never be converted into algorithms. For example, in elementary geometry, there is no algorithm for using figures in an heuristic way (Duval, 1995a) and the way a mathematical proof runs in natural language cannot be formalized but by using symbolic systems. Proofs using natural language cannot be understood by most students (Duval, 1991).

From these observations, we can get a quick outline of the various forms of mathematical processes, as the superposition of a graph on the classification table shows.

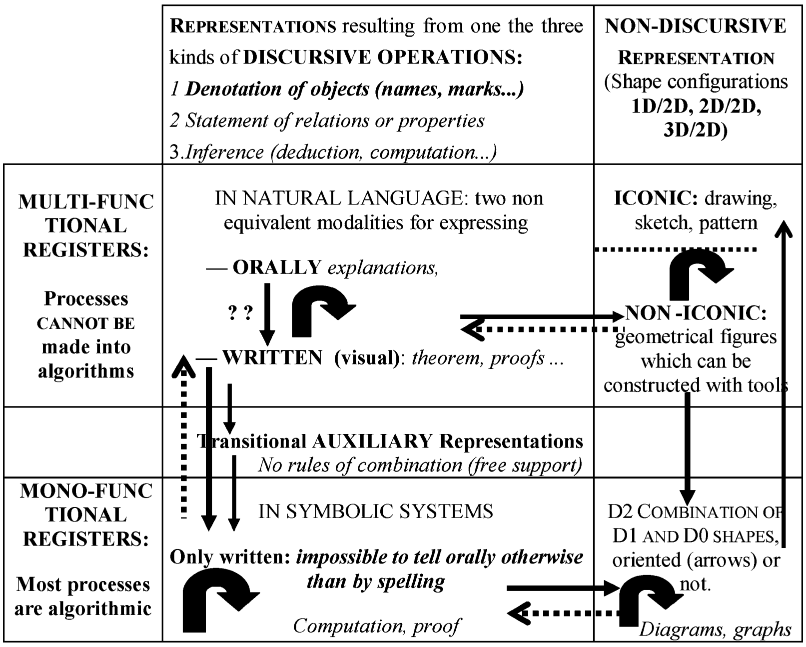


Figure 1. Classification of the registers that can be mobilized in mathematical processes.

What matters for understanding the thinking processes involved in any mathematical activity is to focus on the level of semiotic representation systems and not on the particular representation produced. And the following two points are essential. Firstly, it is only at this level that the basic property of semiotic representation and its significance for mathematics can be grasped: the fact that they can be exchanged one for another, *while keeping the same denotation* (Frege, 1971). Secondly, a mark cannot function as a sign outside of the semiotic system in which its meaning takes on value in opposition to other signs within that system (an example is given later in Figure 15). This idea was the main contribution of Saussure (1973, pp. 158–168) to the analysis of language as a semiotic system. That means, too, that there are rules for producing relevant semiotic representations. Thus, all monofunctional semiotic systems that are characteristic of mathematics are based on rules of representation formation. That can be easily checked for any numeric notation system or for Cartesian graphs.

Of course, some representations that do not depend on a semiotic system are used in mathematical activity. The best example is the matchstick use for representing small integers. They have neither rules of formation nor specific possibilities of transformation. These are used like a material for free manipulations. In that sense, they fit perfectly the third determination of *representamen* given by Peirce: “something that stands *to somebody* . . .”

(1931, p. 2.228). Their use depends only of the *interpretant*. They appear most frequently as transitional auxiliary representations (Hitt, 2003).

Thus, with regard to the property of semiotic representations that is basic for mathematical activity, we can distinguish four very different kinds of semiotic systems. Taking again the word already used by Descartes, in *La Geométrie* (Descartes, 1954, p. 8 (p. 300)), and keeping also its modern meanings we call them "representation *registers*" (Duval, 1995b, p. 21). Not all semiotic systems are registers, only the ones that permit a transformation of representations. We have highlighted the very genuine case of natural language. There, production of semiotic representations can be achieved according to two quite phenomenological modalities. From one to the other there is a big gap, which is very often underestimated (Duval, 2000b).

This classification provides the tools for analyzing mathematical activity and for identifying the root of the troubles with mathematics understanding and not only about such-and-such concept comprehension that many students have.

2.2. *The two types of transformation of semiotic representations*

Insofar as mathematical activity intrinsically consists in the transformation of representations, it becomes obvious that there are two types of transformations of semiotic representations that are radically different: TREATMENTS and CONVERSIONS.

Treatments (curved arrows in the Figure 1) are transformations of representations that happen within the same register: for example, carrying out a calculation while remaining strictly in the same notation system for representing the numbers, solving an equation or system of equations, completing a figure using perceptual criteria of connectivity or symmetry, etc. That gives prominence to the intrinsic role of semiotic systems in mathematical processes. *The treatments, which can be carried out, depend mainly on the possibilities of semiotic transformation, which are specific to the register used.* Two examples suffice to show this.

The procedures for carrying out a numerical operation depend just as much on the system of representation used for the numbers as on the mathematical properties of the operations. Thus, the algorithms are different for a decimal notation and a fractional notation of the same numbers:

$12 + 13 = \ldots$

$0.20 + 0.25 = \ldots \quad 1/5 + 1/4 = \ldots$

$0.20 : 0.25 = \ldots \quad 1/5 : 1/4 = \ldots$

That means that the processes of calculation are never purely mathematical. They depend on the type of representative functioning that the system in use permits. For reasons of economy or visibility one may be led to change notation systems to carry out the treatment.

It is the register of figural transformations of *gestaltist* order that is often called on to solve and justify heuristically many problems of elementary geometry. These transformations are purely visual transformations that can either be carried out simply by changing the vantage point from which they are observed, or be realized materially as if in a jigsaw puzzle. Here are three classical examples where the visual transformations consist of an operation of reconfiguring the original figure (Figure 2).

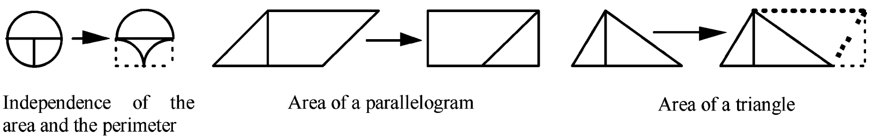


Figure 2. Visual transformations of shapes.

In these examples, the figural units of an original figure can be visually reconfigured without any recourse to a mathematical property. This purely visual operation of reconfiguring an original figure underlies most of the examples of visual evidence that are used in teaching to give "intuitive" explanations of certain mathematical results. But, in most cases it does not work because the visual processes of gestalt recognition do not run in the same way as required and expected from a mathematical point of view (Duval, 1995a).

Conversions (straight arrows in Figure 1) are transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc. Conversion is a representation transformation, which is more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common. It is like a gap that depends on the starting register and the target register (straight arrows in Figure 1). Too often, conversion is classified as translation or encoding. And examples such as the following are put forward (Figure 3).

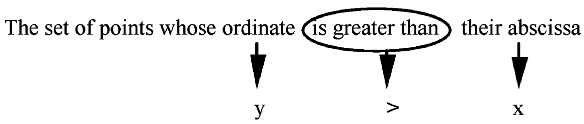


Figure 3. Congruent conversion.

But that is deceptive because a minor modification can cause the rules of encoding or translation to fail (Figure 4).

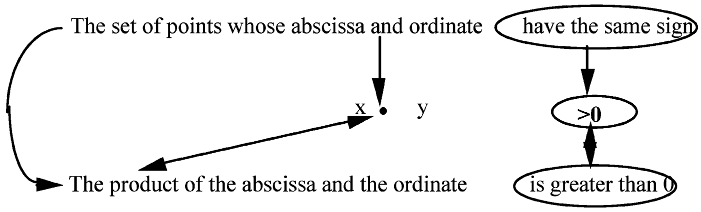


Figure 4. Non-congruent conversion.

Let us now look at a register for which a rule of conversion can be explicitly given. To construct a graph it suffices to have only the following rule: to every ordered pair of numbers one can associate a point on a coordinate plane with given increments on the two axes. And the construction of graphs corresponding to linear functions appears to give students no difficulties whatever. But one has only to reverse the direction of the change of register to see this rule ceases to be operational and sufficient (Figure 5).

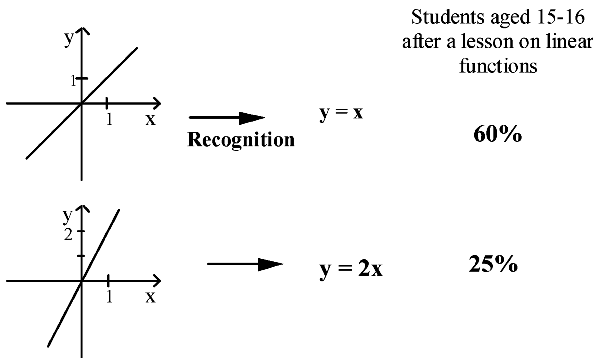


Figure 5. A recognition task.

The task proposed was a task of *simple recognition*, not one of construction or of reading coordinates of points: choose among many possible expressions (for example, among $y = x$, $y = -x$, $y = x + 1$) the one which corresponds to the graph (Duval, 1988). Naturally, if we had asked that the two graphs be constructed the successes would have exceeded 90% in both cases. In standard teaching, the tasks offered are never recognition, but simply reading tasks that require only a process of placing points guided by local understanding and not a process of global interpretation guided by understanding of qualitative visual variables (Figure 15). Converting a semiotic representation into another one cannot be considered either as an encoding or a treatment.

In these two examples, conversion is explicitly required and it appears that it can be confined to transitory situations for solving some particular problem. But most often it is implicitly required whenever two, or even three, registers must be used together in an interactive way. We have already mentioned the case of geometry. There, we are facing something like a hidden gap between the visual process of treatment and the various discursive processes that can be used (Duval, 1998c). And in the classroom we have a very specific practice of simultaneously using two registers. It is spoken in natural language, while it is written in symbolic expressions as if verbal explanations could make any symbolic treatment transparent (Duval, 2000b, pp. 150–155).

Through the various kinds of conversions more than through treatments we touch on the cognitive complexity of comprehension in learning mathematics and on the specific thinking processes required by mathematical activity.

2.3. *How to recognize the same mathematical object through two representations whose contents are heterogeneous?*

In making a distinction, for mathematical signs, between sense and reference Frege (1971, pp. 89, 102–103) emphasized the difference between the content of a representation and what the representation refers to. And between the content of a representation and the represented object there is no other relation than denotation. Now, and this is the decisive consequence that is rarely taken into account, *the content of a representation depends more on the register of the representation than on the object represented* (Duval, 1999, pp. 40–46). That is the reason why passing from one register to another changes not only the means of treatment, but also the properties that can be made explicit. On the other hand, for the non-semiotic representations that are produced by physical devices (mirror, camera, microscope, etc.) or by sensory and brain organizations we have something like a causality relation. The content of a representation is the indirect effect of object. Hence, their "intuitive" or more empirical value (Figure 6).

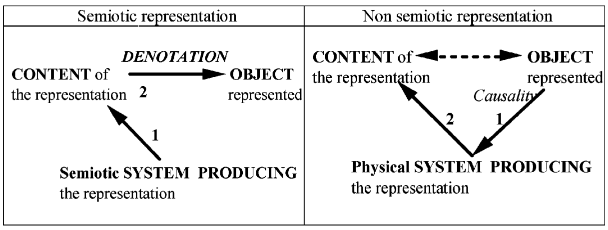


Figure 6. The two kinds of relation between the content of representation and the object represented.

The relation between the content of representation and the object represented depends on the system that is mobilized for producing the representation. We can get iconicity or non-iconicity for semiotic representation as well as for non-semiotic representation. And that brings us back to the cognitive paradox of comprehension in mathematics. How can the represented object be distinguished from the semiotic representation used when there is no access to mathematical object apart from semiotic representations? The first problem of comprehension in learning mathematics is a problem both of recognition and discrimination. When facing two representations from two different registers, how can one recognize the same object represented within their respective content? In other words, how can a student discriminate in any semiotic representation what is mathematically relevant and what is not mathematically relevant? This issue is particularly obvious and crucial for all representations that are produced within multifunctional registers. Does it arise, too, for the representations that are produced within monofunctional registers? In any case, these problems of recognition and discrimination are intrinsic to the construction of connections between registers.

This cognitive paradox makes it possible to put forward the following hypothesis (in mathematical terms "conjecture"): comprehension in mathematics assumes the coordination of at least two registers of semiotic representation. And one can already pose a first question: does such a register coordination come naturally to pupils and students in the context of mathematical teaching?

3. The two sources of incomprehension in the learning of mathematics

The two types of transformation of semiotic representations are quite different sources of recurrent difficulties in learning mathematics. They are not at first difficulties particular to this or that mathematical concept, but rather more global difficulties that can be found at every level of teaching and in every domain of mathematics. For nearly 20 years, empirical data have been collected about the relationships between the thinking processes involved in mathematical activity and troubles of comprehension or even blockages of most learners. And anybody can get empirical evidence on the condition that treatment and conversion be methodologically separated in the tasks that are given to students, which is seldom or never done in most research studies.

We will confine ourselves to giving some examples in order to show the deep misunderstanding of these two types of transformation at different levels of teaching and in the various areas of mathematical activity.

3.1. *A first source of incomprehension: The complexity and specificity of treatments carried out in a multifunctional register*

There is a big source of misunderstanding between teachers and students mainly with regard to basic and complementary thought processes, reasoning and visualization. Unlike monofunctional registers, multifunctional registers seem common and directly accessible to every student. But that is very deceptive. In fact, the mathematical way of using the multifunctional registers runs against the common practice, starting with the practice of natural language (Duval, 1995b, pp. 87–136). We will focus here rather on the figures in geometry insofar as they explicitly resort to visualization and not only to discursive knowledge (properties, definitions, theorems). Recollect that a figure in geometry is always rooted in the functioning of two registers. And if we want to grasp its cognitive complexity, we must analyze separately the way in which the treatments are carried out respectively in the discursive register and the visual register, even though they merge into the same mathematical process. When we focus on visualization we are facing a strong discrepancy between the common way to see the figures, generally in an iconic way, and the mathematical way they are expected to be looked at. There are many ways of "seeing" (Duval, 1995a). Which is the one required by the heuristic use of the figures?

We gave earlier three extremely elementary examples of use of figures in geometry (Figure 2). In these examples, "seeing" consisted of discerning in the original figure the transformations that permit the reconfiguration into the other one: the passage from the original figure to the one which is the goal makes it possible to understand a relation, a computational formula, etc. Thus, assuming the computation of the area of a rectangle to be known, one can see how to compute that of a parallelogram and thence that of a triangle (Figure 7).

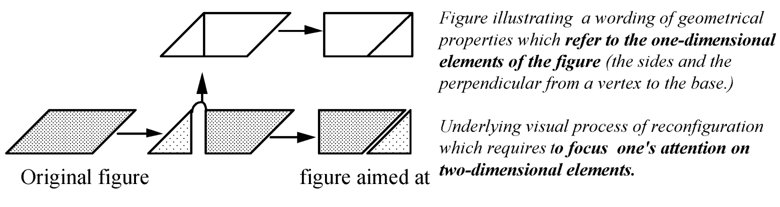


Figure 7. Is the figure illustration cognitively congruent to the visual transformation?

What constitutes the fact of "seeing" in geometry? Although the mathematical discourse necessitates looking at the one-dimensional elements of the figure, the heuristic force of the figure requires that attention be centered on the two-dimensional elements. This example is cited everywhere as a demonstration of a spontaneous activity, which should be common to beginning students and confirmed mathematicians. In reality, the factors that here give the figure its heuristic and explanatory clarity can, in mathematically similar situations, impede seeing, as can be verified in the following example (Figure 8).

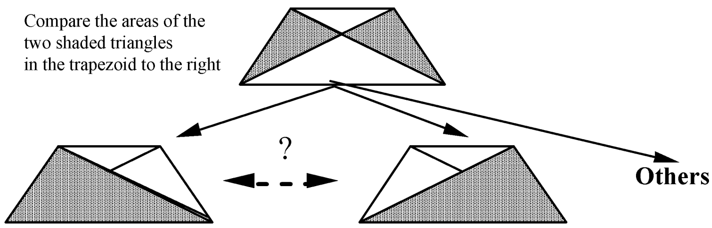


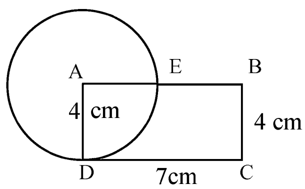
Figure 8. First step of a visual treatment: subfigures required to be discriminated.

The solution of certain problems requires a comparison of certain possible subfigures obtained by reconfiguration and thus the ability to *discern them rapidly* in the original figure. There are factors, which in certain cases facilitate the recognition of the relevant subfigures and inhibit it in others (Duval, 1995a, pp. 144, 149–150). But there are other, possibly more interesting, situations that show the complexity and difficulty of figures: the ones involving a circle and some straight lines. On that subject also we have very dependable observations available, at different levels of teaching.

At the end of elementary school, the presented in Figure 9 problem was given to all French students entering middle school and the data that resulted are the problem in Figure 9. Very often, the same kinds of problem were asked several years running.

On the figure sketched freehand here (the real lengths are written in cm), are represented a rectangle ABCD and a circle with center A, passing through D.

Find the length of segment [EB]



September 1997	A sample of 2604 students	September 1998	2590 students
Mathematical answers **[AE seen as a 4 cm. ray]**	**9%**	Mathematical answers **[AE seen as a 4 cm. ray]**	**22, 2%**
Answers by measuring the segment (about 2 cm. on the segment presented) Answers by visual estimation (E about in the middle of AB, around 3.5)	16% 26%	Answers by measuring the segment (about 3.5 cm. on the segment presented)	39,6%
Other answers No answer	30% 16%	Other answers	24,4%

Figure 9. French national assessment (MEN, 1998, 1999).

In reality, to find the mathematical answer, students had to see within the figure the two subfigures B (see Figure 10) and not the two subfigures A. Because it is only in the two subfigures B that one sees the two rays as a side and a part of the other side of the rectangle. Now it is the subfigures A that leap to the eye and thus tend to screen out the subfigures B!

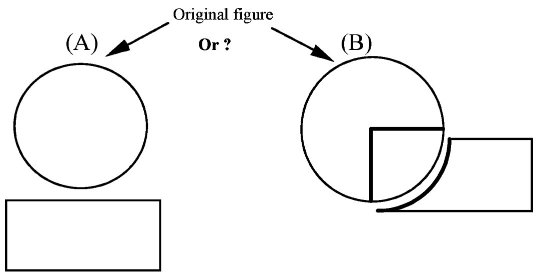


Figure 10. Two ways of identifying subfigures within the original figure.

How does one "see" the original figure in the statement accompanying the statement of the problem (Figure 9)? Most students cannot discriminate the (B) visual organization.

The second survey occurred close to the end of middle school. The following problem was posed (Figure 11).

Is the perimeter of the triangle ABC
larger than
smaller than, or
equal to
the length of the two segments EA and AF?

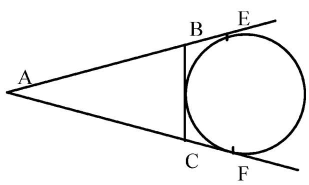


Figure 11. Problem presented to 14-year-olds (Mesquita, 1989, pp. 40, 68–69, 96).

There are two ways of seeing the figure in the statement of the problem, but only one shows the answer and gives the reason (Figure 12).

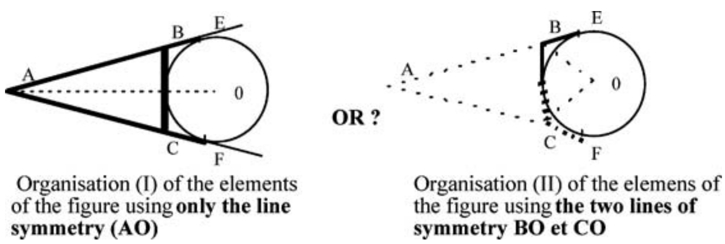


Figure 12. Two figural organizations.

From the figural organization (I) to the figural organization (II) there is a jump, which depends on visual factors. The spontaneous vision that is produced has a single axis of symmetry (organization I), while the solution requires that one give greater importance to two other axes of symmetry (organization II.) Now passing from (I) to (II) constitutes a leap, which more than half of the students did not make. In reality, to be able to see the figure as having two lines of symmetry OB and OC one must break the simple figure element (organization I) consisting of the segment BC into two segments (organization II). And in order to have the majority of the students get to the point of seeing organization (II) in the figure of the statement, the statement of the problem had to be modified by describing the division of segment BC: "let I be the point of intersection of AO and BC; compare BI and IC" (Pluvinage, 1990, p. 27).

These few examples give a good illustration of the complexity of the mathematical use of figures and the non-natural character for most students of the act of seeing in geometry. How should it be analyzed? How should students be introduced to it? As far as observations that can be made in all of the domains of geometry go, two positions are possible.

The first consists of explaining the persistent difficulties that students encounter with figures as misunderstanding of the mathematics represented. Otherwise stated, it would be comprehension of mathematical properties that would guide the reading and exploration of the figures toward the solution of a problem. Good conceptual comprehension ought to lead to seeing in a figure what has to be seen in order to find there the elements for solving a problem.

The second position consists of considering that the figures arise in a system of representation that is independent of the statements and of the mathematical properties to which they refer. That would mean that what one sees in a figure depends on factors of visual organization: it is these factors that determine the discrimination, that is the recognition, of certain one-, two- and three-dimensional forms in a figure and exclude the discrimination of other possible configurations and sub-figures in the same figure. Now "seeing" in geometry frequently requires that one be able to recognize one or another of these other possible configurations and sub-configurations. *What needs to be recognized in an original figure is a function of the statement of the problem, but its "visibility", that is, the more or less spontaneous character of its recognition, depends on visual operations of reorganization.* There are many factors that can inhibit or favor this discrimination of these visual operations. They can be studied experimentally (Duval, 1995a, 1998c; Rommevaux, 1998).

Another observation made by Schoenfeld with older students after a semester of work in geometry shows the independence of figures relative

Take two intersecting lines. Let E be a point on one of them. Construct a circle tangent to the two lines having E as a point of tangency with one of the lines.

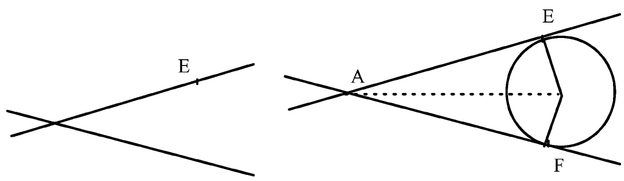


Figure 13. The construction problem posed by Schoenfeld.

to conceptual knowledge and acquired capacities of proving. The construction problem (Figure 13) was proposed to them. The students managed to solve it without much difficulty, but by proceeding entirely empirically. But for them, there was absolutely no connection with all of the mathematical properties they knew on the subject (Schoenfeld, 1986, pp. 243–244, 256).

Note the difference between this construction problem and the preceding problem of comparison of lengths (Figure 11). Success in this construction problem only requires taking into account one axis of symmetry, as in organization (I) of Figure 11. Success in the problem of comparison requires that one recognize the two other axes of symmetry BO and CO, which are “hidden” by the visually predominant AO. Visual recognition does not depend first on conceptual knowledge of properties.

Now we can only mention the important case of language in geometry. We can observe a big gap between a valid deductive reasoning using theorems and the common use of arguments. The two are quite opposite treatments, even though at a surface level the linguistic formulations seem very similar. A valid deductive reasoning runs like a verbal computation of propositions while the use of arguments in order to convince other people runs like the progressive description of a set of beliefs, facts and contradictions. Students can only understand what is a proof when they begin to differentiate these two kinds of reasoning in natural language. In order to make them get to this level, the use of transitional representation activity, such as construction of propositional graphs, is needed (Duval, 1991, 1995b, 1998b).

This first source of difficulty is well known. It gives rise to recurrent observations, which teachers can make, no matter what their level of teaching. It is moreover the reason that in teaching one tends to marginalize, as far as possible, the recourse to multifunctional registers and stay within the monofunctional ones, where treatments can take the form of algorithms. However, the use of natural language cannot be avoided (Duval, 2000b, 2003) and it raises the issue of articulation with the representations produced within the monofunctional registers. And that requires explicit or implicit conversion of representations.

3.2. *A second source of incomprehension: Conversion of representations or change of register.*

Unlike the first, the second type of difficulty has rarely been noticed as such because no sooner do difficulties of conversion appear that they are taken as a sign of conceptual incomprehension. Moreover, to be able genuinely to see the size of the difficulties linked to the conversion of representations, one must set up a mechanism of observation that lets it manifest itself, which assumes for a start that one has become conscious of the difference between treatment and conversion in a mathematical process! In any case, it is this second type of difficulty that limits considerably the capacity of students to use the acquired knowledge as well as their capacity to acquire new knowledge in mathematics. And which very rapidly leads to a limit in progress of comprehension and learning for many students.

The insuperable difficulties raised by conversion can be observed for the different kinds of conversion, which is for each couple of registers to be used together (straight arrow in Figure 1). Thus, the obstacles raised by the simple "translation" of the terms of a word problem into symbolic expressions are also well known. It is a gap that many students cannot succeed getting over, whatever the mathematical content (additive or multiplicative operations on relative numbers, statements to put into equations, etc.). That is the reason why most research has focused on the resorting to transitional auxiliary representations, those spontaneously developed by learners or those to introduce in teaching. In previous studies (1988, 1996b), I gave evidence of a strong failure in converting a Cartesian graph into the corresponding equation. And that failure is quite independent from understanding the concept of function. Figure 5 presents an example of the task of recognition that was used. So we can increase the observations about the conversion troubles for each kind of conversion and in all areas of mathematics teaching. Methodology for that does not at all require only that students be placed in a problem-solving situation or in an application activity. It requires that students be given tasks that are varied systematically not only as a function of the original register but also as a function of internal variations within each register. It can be seen, thus, that it is not just a matter of focusing on errors, which can be observed directly and which recur from one year to another, but that one must dig down to deeper difficulties to be able to analyze problems of comprehension of students of mathematics. When you do that you are facing very deep and amazing phenomena about the cognitive complexity of conversion, in any area of mathematics education.

When you systematically vary a representation within a source register to its converted representation in the target register, you can observe a systematic variation of performances. That happens as if success or systematic mistakes depend on the cognitive distance between the source representation content and the target representation content. In some cases, it is like a one-to-one mapping and the source representation is transparent to the target representation. In these cases, conversion seems nothing more than a simple coding (Figure 3). But in other cases, it no longer runs at all like that (Figure 4). In other words, between a source representation and its converted representation in a target register, there is either congruence or non-congruence. And a more detailed analysis allows us to identify three factors for describing this phenomenon (Duval, 1995b, pp. 49–57):

– A one-to-one mapping between all the meaningful constituents (symbols, words, or visual features) of the contents of the source representation and the target representation is or is not possible.
– The choice for each meaningful constituent of the target representation is or is not univocal.
– For the meaningful constituents that can be mapped, the organization order within the source representation is kept or changed within the target representation.

The second phenomenon is the direction of conversion. When the roles of source register and target register are inverted within a semiotic representation conversion task, the problem is radically changed for students. It can be obvious in one case, while in the inverted task most students systematically fail. It suffices to refer to the example given in Figure 5, recalling that if we had requested the construction of the graphs of the functions $y = x$ and $y = 2x$ or even $y = 1/2x$ there would have been no significant difference in their performances. But the following observation within a domain that seems to give many students difficulties, linear algebra, gives a striking example (Figure 14). Does comprehension in linear algebra not presuppose that students be able to change registers rapidly in an implicit or explicit manner? Would not their difficulty in conversion be one of the major obstacles to surmount? Here, in any case is how one can see the magnitude of this type of difficulty.

We can note the magnitude of the variations in success each time that one reverses the direction of the conversion. Furthermore, not one register considered in isolation appears better mastered than another: performances vary according to the pairs source register, target register. Here we get to the root of trouble in mathematics learning: the ability to understand and to do by oneself any change of representation register. The troubles that many students have with mathematical thinking lie in the mathematical specificity and the cognitive complexity of conversion and changing representation. It is neither a matter of coding nor a matter of mathematical concept alone.

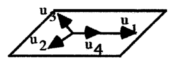
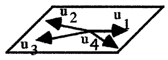
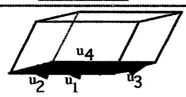
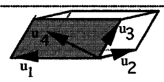
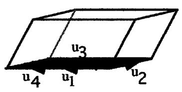
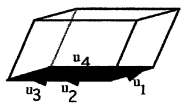
	Questions (March 1993)		Success levels
	Original register	**Destination register**	Sample size: 144
T2→ G	$\begin{bmatrix}1&0&k&p\\0&1&m&0\end{bmatrix}$		**.83**
G → T2		$\begin{bmatrix}1&0&a&c\\0&1&b&d\end{bmatrix}$	**.34**
T3→ G	$\begin{bmatrix}1&k&0&m\\0&0&1&n\\0&0&0&0\end{bmatrix}$		**.68**
G → T3		$\begin{bmatrix}1&0&0&a\\0&1&0&0\\0&0&1&b\end{bmatrix}$	**.35**
T → S	$\begin{bmatrix}1&0&k&p\\0&1&m&0\end{bmatrix}$	$u_1 \in R^2, u_2 \in R^2, u_3 \in R^2, u_4 \in R^2$ $u_1 = 1u_1 + 0u_2$ $u_2 = 0u_1 + 1u_2$ $u_3 = ku_1 + mu_2$ $u_4 = pu_1 + 0u_2$	**.07**
S → T	$u_1 \in R^2, u_2 \in R^2, u_3 \in R^2, u_4 \in R^2$ $u_1 = 1u_1 + 0u_3$ $u_2 = ku_1 + 0u_3 ; k \in R$ $u_3 = 0u_1 + 1u_3$ $u_4 = au_1 + bu_3 ; a \in R, b \in R$	$\begin{bmatrix}1&k&0&a\\0&0&1&b\end{bmatrix}$	**.72**
G → S		$u_1 \in R^3, u_2 \in R^3, u_3 \in R^3, u_4 \in R^3$ $u_1 = 1u_1 + 0u_2$ $u_2 = 0u_1 + 1u_2$ $u_3 = ku_1 + mu_2 ; k \in R, m \in R$ $u_4 = nu_1 + 0u_2 ; n \in R$	**.05**
S → G	$u_1 \in R^3, u_2 \in R^3, u_3 \in R^3, u_4 \in R^3$ $u_1 = 1u_1 + 0u_2$ $u_2 = 0u_1 + 1u_2$ $u_3 = 0u_1 + ku_2 ; k \in R$ $u_4 = au_1 + bu_2 ; a \in R, b \in R$		**.40**

Figure 14. A recognition task (Pavlopoulou, 1993, p. 84).

This complexity appears through two phenomena, of which variation depends on the nature of the two registers mobilized for a representation transformation: the variability of congruence/non-congruence for representations of the same knowledge object and the non-reversibility. In fact, whatever the level and whatever the area, the non-congruent conversions are for many students an impassable barrier in their mathematics comprehension and therefore for their learning.

Facing non-congruent representation conversion, learners are trapped in a conflict between mathematical knowledge requirement and cognitive impossibility:

– Conversion of representation requires the cognitive DISSOCIATION of the represented object and the content of the particular semiotic representation through which it has been first introduced and used in teaching.
– But there is a cognitive IMPOSSIBILITY OF DISSOCIATING any semiotic representation content and its first represented object when there is no other possible access to mathematical object than semiotic.

That conflict leads to the consideration of two representations of the same object as being two mathematical objects. The consequence is then the inability to change register and to use knowledge outside of narrow learning contexts. The registers of the representations remain compartmentalized, and only fragmentary and monoregistral comprehension is possible. Under what conditions can learners be enabled to do such dissociation?

3.3. *How to discriminate in any representation content, whatever the register used, what is mathematically relevant and what is not?*

Herein, obviously lies the more crucial issue for mathematics learning.

Let us take the elementary instance of the linear functions that we have given (Figure 5). Watching their algebraic expression and their graph together, or knowing how to plot their graph from their algebraic expression, is not at all enough to recognize the same function through these two kinds of representation. A deeper cognitive condition is needed: being able to discern *how two graphs that seem visually alike are mathematically different.* When they are taken two by two, they visually contrast by one or several visual features. When they contrast by two (or more) visual features, these are merged as if it were only one. Visual discrimination of graphs is nothing obvious, particularly when they seem very similar in form and content. In fact, the ability to discriminate what is mathematically relevant in each one depends on the implicit construction of such a cognitive network as in the following Figure 15.

In this network, each visual feature matches a symbol category of algebraic expression $y = ax + b$. By "symbol category" we mean a qualitative opposition ($a > 1$ $a < 1$ $a = 1$ or $a = -1$) and not merely numerical variation ($a = 1.65$ or $a = 2.3$). Such a network can be extended to all kinds of function representation and to representations of relations that are not functions (Duval, 1993, p. 46).

How can one help students realize all these representation discriminations within the same register? Here we must pay attention to a very important fact. We have as many visual representations as we want, but not all of them are relevant from a mathematical point of view. Furthermore, not all numerical value variations (here of linear functions) are

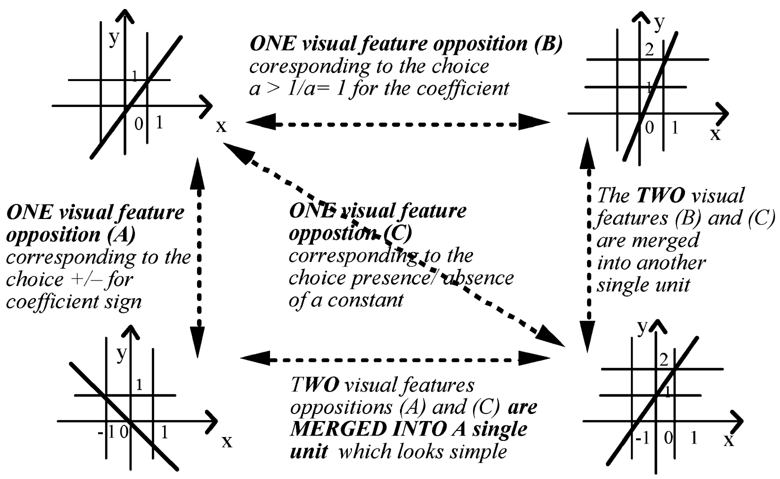


Figure 15. Early connections of a cognitive network for any graphic representation discrimination.

significant for realizing this cognitive network. In order to make students notice the basic visual features *oppositions that are mathematically relevant and cognitively significant*, any representation discrimination task has to be integrated into a conversion task. It is only by investigating representation variations in the source register and representation variations in a target register, that students can at the same time realize what is mathematically relevant in a representation, achieve its conversion in another register and dissociate the represented object from the content of these representations.

We have taken a very elementary example that is very simple to analyze because conversion there occurs between two monofunctional registers, the one non-discursive (graphs) and the other discursive (algebraic writing of relations). But the analysis method used in this particular example goes for every kind of conversion (Figure 1), even for the most complex ones when cognitive distance is becoming large, as between multifunctional register (mother tongue, natural language) and monofunctional register (symbolic system). And at least in its implicit modality, this kind of conversion is continuously needed in teaching where we have always a double semiotic production: oral speech for giving explanations in common language and symbolic or diagrammatic writing for mathematical treatment (Duval, 2000b, pp. 152–155). The most surprising is that transitional auxiliary representations, even the most iconic or concrete ones, need also to be integrated with systematic co-variation tasks if we want them to be efficient!

From that example we can get a glimpse of the specific thinking processes that are required in mathematics. Not only do they use semiotic representation systems but also above all they require their cognitive coordination. And for an obvious reason, a double semiotic access must compensate for the cognitive limitation of the lack of a real double access. That means dissociation between representation content and represented object necessarily involves COORDINATION between different representation registers. Mathematical comprehension begins when coordination of registers starts up. Recognition of the same mathematical objects through representations from two different registers is not a local or occasional operation, but the outcome of global register coordination. Mathematical thinking processes depend on a cognitive synergy of registers of representation. Coordination of registers of semiotic representations provides something like an extension of mental capacity. In this perspective, the opposition often made between comprehension as being conceptual or purely mental and semiotic representations as being external appears to be a deceptive opposition. In fact, mental representations that are useful or pertinent in mathematics are always interiorized semiotic representations.

4. Conclusion

When we analyze mathematical activity from a cognitive point of view three specific characteristics, closely connected, must be taken into account:

(1) It runs through a transformation of semiotic representations, which involves the use of some semiotic system.
(2) For carrying out this transformation, quite different registers of semiotic representations can be used.
(3) Mathematical objects must never be confused with the semiotic representations used, although there is no access to them other than using semiotic representation.

Thus, it appears that the thinking processes in mathematics are based on two quite different kinds of transformations of representations. Even if a single representation register is enough from a mathematical point of view, from a cognitive point of view mathematical activity involves the simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another. In other words, conceptual comprehension in mathematics involves a two-register synergy, and sometimes a three-register synergy. That is the reason why what is mathematically simple and occurs at the initial stage of mathematical knowledge construction can be cognitively complex and requires a development of a specific awareness about this coordination of registers.

The distinction among four kinds of representation registers highlights the variety and the cognitive gap of representation conversion according to the source register and the target register. It also makes it possible to define some variables for analyzing the cognitive complexity underlying any mathematical activity, either for a research aim or for an education aim. And the distinction between multifunctional and monofunctional registers shows how, for all the transformations that are treatments, visualization and language can be used in quite different ways than the usual way within the other areas of knowledge and in everyday life. Practices of these registers that the students may have outside of mathematics seem often to screen out the manner in which they should be mobilized in mathematics. That raises a deep ambiguity in teaching: on one hand, these registers are avoided because students have a great deal of difficulty carrying out mathematical processes there, and on the other hand, they are used for giving "meaning" to mathematical processes that are carried out within monofunctional registers. In teaching, we can observe quite opposite practices of these multifunctional registers.

It is within the framework of such a cognitive model of mathematical thinking processes that we can analyze in depth the obstacles to mathematics comprehension. Treatments, mainly within multifunctional registers, and conversions are quite independent sources of incomprehension. But the root of the troubles that many students have with mathematical thinking lies in the mathematical specificity and the cognitive complexity of conversion and changing representation. We cannot deeply analyze and understand the problem of mathematics comprehension for most learners if we do not start by separating the two types of representation transformation. This is rarely, if ever, done, either because conversion is judged to be a type of treatment or because it is believed to depend on conceptual comprehension, that is, a purely "mental", i.e., asemiotic, activity. And there are always good reasons for that.

In the first place, from a mathematical point of view, conversion comes in solely for the purpose of choosing the register in which the necessary treatments can be carried out most economically or most powerfully, or of providing a second register to serve as a support or guide for the treatments being carried out in another register. In other terms, conversion plays no intrinsic role in mathematical processes of justification or proof. Because these are achieved on the basis of a treatment that is carried out within a single register, mainly a discursive one and most often some monofunctional register. In fact, conversion cannot be separated from treatment because it is the choice of treatment that makes the choice of register relevant. In the second place, research in mathematics education is almost always carried out on the ways of teaching particular conceptual contents and procedures for each level of curriculum. What concerns mathematical activity is pushed back into the background or explained either by conceptual understanding (or misunderstanding) or by a common pedagogical framework about the importance of student' activity and the role of their mental representations for comprehension. This leads to wiping out the importance of the diversity of representation registers and to acting as if all representations of the same mathematical object had the same content or as if content of one could be seen from another as if by transparency. In other words, some isomorphism between representations from two different semiotic systems or between processes that are performed within two semiotic systems is implicitly assumed. Recollect that Piaget made this search for isomorphisms one of the key principles of an analysis of the development of knowledge in children, even though, later, he limited himself to the search for "partial isomorphisms" (Piaget, 1967, pp. 73–74, 262–266) and great theoretical use was made of them in the analysis of genetic epistemology as in certain didactical studies. But does mathematical isomorphism involve the cognitive isomorphism between the semiotic representations used? Pushing back into the background the three specific characteristics mentioned earlier confines most students in what has been described as "compartmentalization" of mathematical knowledge.

Changing representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum. It depends on coordination of several representation registers and it is only in mathematics that such a register coordination is strongly needed. Is this basic requirement really taken into account? Too often, investigations focus on what the right representations are or what the most accessible register would be in order to make students truly understand and use some particular mathematical knowledge. With such concern of this type teaching goes no further than a surface level. What will the students do when they are confronted by quite other representations or different situations? Even auxiliary and individual representations, the most iconic or concrete ones, need to be articulated with the semiotic representations produced within semiotic systems. The true challenge of mathematics education is first to develop the ability to change representation register.

NOTES

1. A first sketch of this paper has been presented in Mediterranean Journal for Research in Mathematics Education 2002, 1, 2, 1–16. We present here a more developed version of the cognitive model of mathematical activity and thinking.

2. The relation of a subject to an object is the basic epistemological distinction for analysing knowledge (Kant, 1956, p. 63, 296; Piaget, 1967, p. 65 and 1973, p. 31). Thus "object" can be used with three different meanings:

   (1) *the invariant of a set of phenomena or the invariant of some multiplicity of possible representations*. In that sense "objects" are KNOWLEDGE OBJECTS.
   (2) *the target of attention focusing on such or such aspect* (shape, position, size, succession...) of what is given. In that sense "objects' are transient PHENOMENOLOGICAL OBJECTS.
   (3) *the data given by perception, or the physical things*. In that sense, "objects" are CONCRETE OBJECTS.

   Mathematical objects (numbers, functions, vectors, etc.) are knowledge objects, and semiotic representations which can support two quite opposite foci of attention (either the visual data given or some represented object which can be a concrete one or some invariant) are transient phenomenological objects. If we consider an algebraic equation and the graph of a line, they are first different semiotic representations. They are "mathematical objects" under the condition that attention can focus on some invariant (the assumed represented relations) and not only on their visual data and their perceptual organization (Duval, 1995b, pp. 53–54; 2002). It is only from a strict formal point of view that semiotic representations can be taken as concrete objects (Duval, 1998a, pp. 160–163).

REFERENCES

Descartes, R.: 1954, *The Geometry of Rene Descartes* (translated from French and Latin), Dover, New York.

Duval, R.: 1983, 'L'obstacle du dédoublement des objets mathématiques', *Educational Studies in Mathematics* 14(4), 385–414.

Duval, R.: 1988, 'Graphiques et équations: l'articulation de deux registres', *Annales de Didactique et de Sciences Cognitives* 1, 235–253.

Duval, R.: 1991, 'Structure du raisonnement déductif et apprentissage de la Démonstration', *Educational Studies in Mathematics* 22(3), 233–261.

Duval, R.: 1993, 'Registres de représentation sémiotique et fonctionnement cognitif de la pensée', *Annales de Didactique et de Sciences Cognitives* 5, 37–65.

Duval, R.: 1995a, 'Geometrical Pictures: Kinds of representation and specific processing', in R. Suttherland and J. Mason (eds.), *Exploiting Mental Imagery with Computers in Mathematics Education*, Springer, Berlin, pp. 142–157.

Duval, R.: 1995b, *Sémiosis et pensée humaine*, Berne, Peter Lang.

Duval, R.: 1996a, 'Quel cognitif retenir en didactique des mathématiques?' *Recherches en Didactique des Mathématiques* 16(3), 349–382.

Duval, R.: 1996b, 'Les représentations graphiques: fonctionnement et conditions de leur apprentissage', in *Actes de la 46ème Rencontre Internationale de la CIEAEM,* tome 1, 3–15, Université Paul Sabatier, Toulouse, pp. 3–15.

Duval, R.: 1998a, 'Signe et objet (I): trois grandes étapes dans la problématique des rapports entre représentation et objet', *Annales de Didactique et de Sciences Cognitives* 6, 139–163.

Duval, R.: 1998b, 'Geometry from a cognitive point a view', in C. Mammana and V. Villani (eds.), *Perspectives on the Teaching of Geometry for the 21st Century*, Kluwer Academic Publishers, Dordrecht, pp. 37–52.

Duval, R.: 2000a, 'Basic issues for research in mathematics education', in T. Nakahara and M. Koyama (eds.), *Proceedings of the 24th Conference of PME*, 1, Nishiki Print Co. Ltd., Hiroshima, pp. 55–69.

Duval, R.: 2000b, 'Ecriture, raisonnement et découverte de la démonstration en mathématiques', *Recherches en Didactique des Mathématiques* 20(2), 135–170.

Duval, R.: 2002, 'L'apprentissage de l'algèbre et le problème cognitif de la désignation des objets', in J.Ph. Drouhard and M. Maurel (eds.), *Actes des Séminaires SFIDA*, 13–16 (IV), IREM, Nice, pp. 67–94.

Duval, R.: 2003, 'Langage(s) et représentation(s) dans l'enseignement des mathématiques: deux pratiques et une troisième', in M. Kourkoulos, G. Toulis and C. Tzanakis (eds.), *Proceedings 3rd Colloquium on the Didactics of Mathematics*, University of Crete, Rethymnon, pp. 13–33.

Duval, R. (ed.): 1999, *Conversion et articulation des représentations analogiques*, Séminaire I.U.F.M., D.R.E.D., Villeneuve d'Ascq.

Hitt, F. (ed.), 2002, *Representations and Mathematics Visualization*, North American Chapter of IGPME, Cinvestav-IPN, Mexico.

Hitt, F.: 2003, 'Le caractère fonctionnel des représentations', *Annales de Didactique et de Sciences Cognitives* 8, 255–271.

Frege, G.: 1971, 'Sens et dénotation', in *Ecrits logiques et philosophiques* (translated by C. Imbert), Seuil, Paris.

Kant, E.: 1956, *Kritik der reinen Vernunft*, Felix Meiner, Hamburg.

MEN: 1997, 'Évaluation CE2-6ème Résultats nationaux-Septembre 1996', *Les Dossiers* 79, Ministère de l'Education Nationale, Paris.

MEN: 1998, 'Évaluation CE2-6ème Repères nationaux-Septembre 1997', *Les Dossiers* 100, Ministère de l'Education Nationale, Paris.

MEN: 1999, 'Évaluation CE2-6ème Repères nationaux-Septembre 1998', *Les Dossiers* 111, Ministère de l'Education Nationale, Paris.

Pavlopoulou, K.: 1993, 'Un problème décisif pour l'apprentissage de l'algèbre linéaire: la coordination des registres de représentation', *Annales de Didactique et de Sciences Cognitives* 5, 67–93.

CP = Peirce, C.S.: 1931, *Collected Papers*, *II, Elements of Logic*, Harvard University Press, Cambridge, Massachusetts.

Piaget, J.: 1923, *Le langage et la pensée chez l'enfant,* Delachaux et Niestlé,Neuchâtel.

Piaget, J.: 1926, *La représentation du monde chez l'enfant*, Alcan, Paris.

Piaget, J.: 1967, *Biologie et connaissance*, Gallimard, Paris.

Piaget, J.: 1973, *Introduction à l'épistémologie génétique, 1, La pensée mathématique*, P.U.F., Paris.

Pluvinage, F.: 1990, 'Didactique de la résolution de problèmes', *Annales de Didactique et de Sciences Cognitives* 3, 7–34.

Rommevaux, M.P.: 1998, 'Le discernement des plans dans une situation tridimensionnelle', *Annales de Didactique et de Sciences Cognitives* 6, 27–65.

Saussure (de), F.: 1973 (1915), *Cours de linguistique générale,* Payot, Paris.

Schoenfeld, A.H.: 1986, 'On having and using Geometric Knowledge', in J. Hiebert (ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*, Lawrence Erlbaum Associates, Hillsdale, NJ, pp. 225–264.

Vygotski, L.S.: 1985, *Pensée et langage,* Editions Sociales, Paris.

RAYMOND DUVAL
Laboratoire Mutations des Systèmes Educatifs
Université du Littoral
Maison de la Recherche
17 rue du Puits d'Amour B.P. 751
62231 Boulogne-sur-Mer Cedex
France
E-mail: duval@wanadoo.fr