BIT 4 (1964), 205–212

GENERATION OF RANDOM BIVARIATE NORMAL DEVIATES AND COMPUTATION OF RELATED INTEGRALS

BIRGER JANSSON

Abstract.

A method which transforms two random variables having rectangular distributions into a pair of bivariate normal deviates with prescribed covariance matrix is described. The same transformation is used for integrating the bivariate normal distribution over areas which are the intersection of the domain outside an equiprobability ellipse and a sector determined by two lines through the point of gravity of the normal distribution.

1. Introduction.

In Annals of Math. Stat. 1958 G. E. P. Box and Mervin E. Muller [1] introduced a method of generating random normal deviates from a distribution with mean 0 and standard deviation 1. Their method, which gives for each generation two independent normal deviates, may be generalized so that the two generated numbers belong to a specified bivariate normal distribution.

2. Method.

Let u and v be two random variables of a rectangular distribution on the interval $(0,1]$. Determine x and y by

$$\left.\begin{aligned} x &= \mu_x+\sigma_x\cdot\sqrt{-2\log u}\cdot\left(\sqrt{1-\varrho^2}\cdot\cos 2\pi v+\varrho\sin 2\pi v\right) \\ y &= \mu_y+\sigma_y\cdot\sqrt{-2\log u}\cdot\sin 2\pi v \end{aligned}\right\} \qquad (1)$$

Then x and y belong to a bivariate normal distribution with means μ_x and μ_y and standard deviations σ_x and σ_y of x and y respectively and with the correlation coefficient ϱ.

3. Proof.

An inversion of (1) gives

$$\left.\begin{aligned} u &= \exp\left\{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\varrho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\} \\ v &= \frac{1}{2\pi}\operatorname{arctg}\left(\frac{\sqrt{1-\varrho^2}}{\frac{x-\mu_x}{\sigma_x}\cdot\frac{\sigma_y}{y-\mu_y}-\varrho}\right) \end{aligned}\right\} \quad (2)$$

from which the Jacobian J may be derived. We get

$$J = \frac{-1}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}}\exp\left\{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\varrho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\} \quad (3)$$

A probability element $dudv$ in the u,v-domain is transformed by (1) into $|J|\cdot dxdy$ in the x,y-domain where J is given by (3), which completes the proof.

With $\mu_x=\mu_y=\varrho=0$ and $\sigma_x=\sigma_y=1$ we get from (1)

$$\left.\begin{aligned} x &= \sqrt{-2\log u}\cdot\cos 2\pi v \\ y &= \sqrt{-2\log u}\cdot\sin 2\pi v \end{aligned}\right\} \quad (4)$$

which is Box–Muller's formulas for generation of a pair of independent normal deviates.

4. Accuracy.

When the proposed method is used for generation of bivariate normal deviates on a computer the accuracy depends partly on the accuracy of the calculation of trigonometric numbers and logarithms on the computer, and partly on how good the generation of the rectangular random numbers is. A discussion of how to evaluate some random number generators, which produce rectangular random numbers, was given by the author in [2].

5. Generation of random variables from distributions related to the normal distribution.

As already pointed out by Box–Muller [1] the formulas (4) may be used to generate random numbers from distributions derived from the one-dimensional normal distributions. Thus we have for example

a) *n-dimensional normal variables.* Using (4) $\frac{n}{2}$ or $\frac{n+1}{2}$ times for n even and odd respectively, we obtain n independent normal deviates (when n

is odd we get one normal deviate further which is not used). With knowledge of the desired covariance matrix, and with standard transformations we obtain the variables wanted.

b) *χ_n^2-variables.* If n is the number of degrees of freedom of the χ^2-distribution we get, using the definition of χ_n^2 as the sum of n squares of independent normal deviates with mean 0 and variance 1:

$$\begin{aligned} x_{2k} &= -2\log(u_1 \cdot u_2 \cdot \ldots \cdot u_k) \\ x_{2k+1} &= -2\log(u_1 \cdot u_2 \cdot \ldots \cdot u_k) - 2\log u_{k+1} \cdot \cos^2 2\pi u_{k+2} \end{aligned} \tag{5}$$

where $u_1, u_2, \ldots$ are rectangularly distributed on $(0,1]$ and x_{2k} and x_{2k+1} are χ^2-distributed with $2k$ and $2k+1$ degrees of freedom respectively.

c) *Variables from the t-, F- or β-distributions.* From the definitions of these three distributions we get

$$t = \frac{x}{\sqrt{x_n/n}} \tag{6}$$

$$F = \frac{x_m/m}{x_n/n} \tag{7}$$

$$\beta = \frac{x_m}{x_m + x_n} \tag{8}$$

where x is normal and obtained by (4) and x_n and x_m are χ^2-distributed and obtained by (5) with n and m degrees of freedom respectively.

6. Some integral formulas.

The transformation (1) may be used in context with integration of the bivariate normal frequency function over certain areas in the x,y-domain. Let us first determine which area in the x,y-domain corresponds to a rectangular area $0 \leqq u \leqq u'$, $0 \leqq v \leqq v'$ in the u,v-domain (fig. 1).

Denote the frequency function by $f(x,y)$ i.e.

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}} \exp\left\{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\varrho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\} \tag{9}$$

Directly from the first equation in (2) it follows that $u \leqq u'$ corresponds to $f(x,y) \leqq \dfrac{u'}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}}$ which is valid for the area outside the equiprobability ellipse

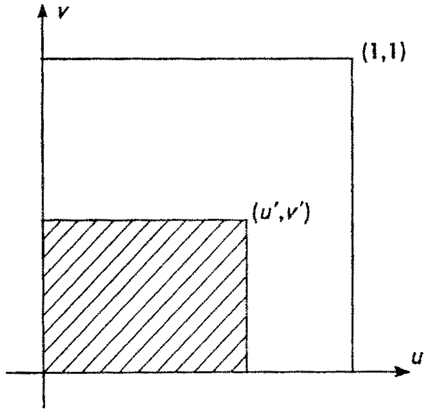

Fig. 1

$$g(x,y) = \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\varrho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 + 2(1-\varrho^2)\cdot\log u' = 0 \tag{10}$$

When $u' \to 0$ the area of the ellipse tends to ∞ and when $u' \to 1$ the area of the ellipse approaches 0, i.e. the ellipse degenerates to the point $x=\mu_x$, $y=\mu_y$ (fig. 2).

The second equation in (2) may be written more conveniently

$$\frac{y-\mu_y}{\sigma_y} = \frac{\sin 2\pi v}{\sqrt{1-\varrho^2}\cdot\cos 2\pi v + \varrho \sin 2\pi v} \cdot \frac{x-\mu_x}{\sigma_x} \tag{11}$$

representing a family of lines through the center of gravity (μ_x, μ_y).

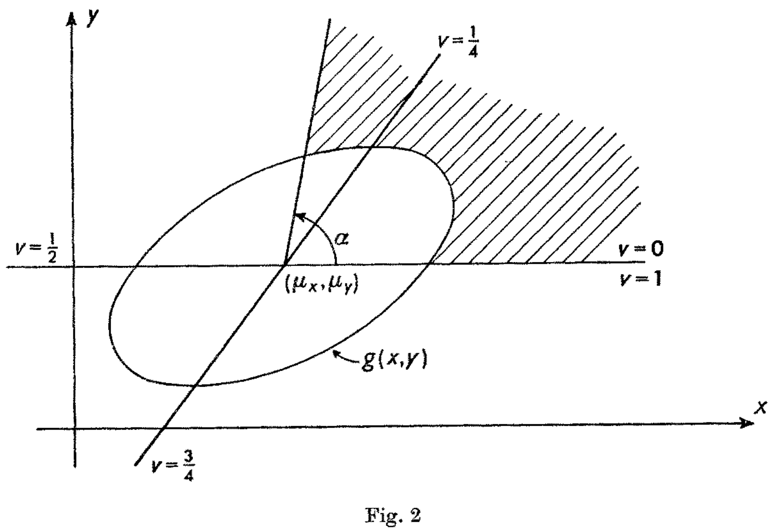

Fig. 2

Just regarding half-lines starting from (μ_x, μ_y) we may see that

$$v = 0 \text{ corresponds to } y = \mu_y \text{ for } x > \mu_x$$

$$v = \tfrac{1}{2} \text{ corresponds to } y = \mu_y \text{ for } x < \mu_x$$

$$v = 1 \text{ corresponds to } y = \mu_y \text{ for } x > \mu_x$$

$$v = \tfrac{1}{4} \text{ corresponds to } \frac{y-\mu_y}{\sigma_y} = \frac{1}{\varrho}\cdot\frac{x-\mu_x}{\sigma_x} \text{ for } y > \mu_y$$

which is one half of the regression line of x on y while

$v = \tfrac{3}{4}$ corresponds to the second half of the same regression line.

Define an angle α in the interval $[0,\pi)$ according to fig. 2. It then follows from (11) that

$$\operatorname{tg}\alpha = \frac{\sigma_y}{\sigma_x}\cdot\frac{\sin 2\pi v}{\sqrt{1-\varrho^2}\cdot\cos 2\pi v+\varrho\sin 2\pi v} \tag{12}$$

whenever $0 \leqq x < \tfrac{1}{2}$. It follows from (12) that α increases from 0 to π when v increases from 0 to $\tfrac{1}{2}$.

When $\tfrac{1}{2} \leqq v < 1$ we get by (12) an angle α_1 in $[0,\pi)$ and then the angle α in $[\pi, 2\pi)$ by

$$\alpha = \pi + \alpha_1 . \tag{13}$$

When $v=1$ we finally get $\alpha = 2\pi$.

It is thus clear that the interval $0 \leqq v \leqq v'$ transforms into the area between the two half-lines corresponding to $v=0$ and $v=v'$.

We have thus found that the shaded rectangular area in fig. 1 is transformed by the transformation (1) into the shaded area in fig. 2, which is the intersection between the area outside $g(x,y)=0$ according to (10) and the area between the half-lines determined by (12) and (13).

By inverting formula (12) we get

$$v = \frac{1}{2\pi}\operatorname{arctg}\frac{\sigma_x\sqrt{1-\varrho^2}\cdot\operatorname{tg}\alpha}{\sigma_y-\sigma_x\cdot\varrho\cdot\operatorname{tg}\alpha} \tag{14}$$

which is valid for $0 \leqq \alpha < \pi$.

If $\pi \leqq \alpha < 2\pi$ we see from (12) and (13) that we have to add $\tfrac{1}{2}$ to the v-value obtained by (14) and finally when $\alpha = 2\pi$ we have $v=1$.

According to the rectangular distribution of u and v the mass over $0 \leqq u \leqq u'$, $0 \leqq v \leqq v'$ is $u'v'$ and so we may conclude with the following integral formula:

THEOREM. If the area $A(\alpha^*, K)$ is defined by

1) $f(x,y) \leqq K$ with K in the interval $[0, (2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2})^{-1}]$.
2) $0 \leqq \alpha \leqq \alpha^* \leqq 2\pi$

we have

$$\iint_{A(\alpha^*, K)} f(x,y)\,dx\,dy = \sigma_x\sigma_y\sqrt{1-\varrho^2}\cdot K\left(\pi\cdot\delta + \operatorname{arctg}\frac{\sigma_x\sqrt{1-\varrho^2}\cdot\operatorname{tg}\alpha^*}{\sigma_y - \sigma_x\varrho\cdot\operatorname{tg}\alpha^*}\right)$$

where

$$\delta = \begin{cases} 0 \text{ when } & 0 \leqq \alpha^* < \pi \\ 1 \text{ when } & \pi \leqq \alpha^* < 2\pi \\ 2 \text{ when } & \alpha^* = 2\pi \end{cases}$$ and where the arctg is to be chosen in $[0, \pi)$.

This paper may be finished with two corollaries which are direct consequences of the theorem.

COROLLARY 1. With $\alpha^* = 2\pi$ we get the mass over the area for which $f(x,y) \leqq K$, i.e. outside an equi-probability ellipse in the x,y-plane. We get

$$\iint_{A(2\pi, K)} f(x,y)\,dx\,dy = 2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}\cdot K\,.$$

When the ellipse degenerates to a point we have

$$K = f(\mu_x, \mu_y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}}$$

which gives

$$\iint_{R_2} f(x,y)\,dx\,dy = 1$$

where R_2 is the full x,y-plane.

COROLLARY 2. With $\alpha^* = \frac{\pi}{2}$ we get the mass over an area of the x,y-plane for which $x \geqq \mu_x$, $y \geqq \mu_y$ and $f(x,y) \leqq K$. We then get

$$\iint_{A\left(\frac{\pi}{2}, K\right)} f(x,y)\,dx\,dy = \sigma_x\sigma_y\sqrt{1-\varrho^2}\cdot K\cdot\operatorname{arctg}\left(-\frac{\sqrt{1-\varrho^2}}{\varrho}\right) =$$

$$= \sigma_x\sigma_y\sqrt{1-\varrho^2}\cdot K\cdot(\pi - \arccos\varrho)\,.$$

Further, if we put $K = \dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}}$, i.e. we integrate over the quadrant where $x \geqq \mu_x$ and $y \geqq \mu_y$ we get

$$\int_{\mu_y}^{\infty}\int_{\mu_x}^{\infty} f(x,y)\,dx\,dy = \frac{1}{2} - \frac{1}{2\pi}\arccos\varrho \; .$$

This formula is usually expressed with the arcsine-function (see e.g. Cramér, [3], p. 290). In this paper, however, we have defined the angle in the interval $[0,\pi)$, in which arctg and arccos are one-valued functions but arcsin a two-valued function and this consequently determines the choice of arctg or arccos.

In the half-plane determined by $\alpha^* = \pi$ we have the mass $\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}K$ which follows from corollary 1. In the quadrant defined by $x \leqq \mu_x$, $y \geqq \mu_y$ we thus get

$$\iint\limits_{A(\pi,K)-A\left(\frac{\pi}{2},K\right)} f(x,y)\,dx\,dy = \pi\sigma_x\sigma_y\sqrt{1-\varrho^2}K - \sigma_x\sigma_y\sqrt{1-\varrho^2}K(\pi - \arccos\varrho)$$

$$= \sigma_x\sigma_y\sqrt{1-\varrho^2}K\cdot\arccos\varrho$$

which all over the quadrant $\left(\text{i.e. with } K = \dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\varrho^2}}\right)$ gives

$$\int_{\mu_y}^{\infty}\int_{-\infty}^{\mu_x} f(x,y)\,dx\,dy = \frac{1}{2\pi}\arccos\varrho \; .$$

The masses in the third and fourth quadrants are the same as the corresponding masses in the first and second quadrants respectively.

REFERENCES

1. G. E. P. Box and Mervin E. Muller, *A Note on the Generation of Random Normal Deviates*, Annals Math. Stat. 29 (1958), pp. 610–611.
2. Birger Jansson, *Autocorrelations between pseudo-random numbers*, BIT no. 1, 1964.
3. Harald Cramér, *Mathematical Methods of Statistics*, Princeton 1946.

Besides these three references, referred to in the text, some papers further may be mentioned. The method, proposed by Marsaglia, to generate random variables from other distributions than the uniform one using uniform random variables is especially interesting due to the fact that it is faster than other proposed methods. The computer memory space requirements of the Marsaglia method are, however, considerable. The papers by Gupta surveys the literature on integrals of multivariate normal distributions and contains a very comprehensive bibliography.

4. J. C. Butcher, *Random sampling from the normal distribution*, Comp. J. 3 (1961), pp. 251–253.
5. G. Marsaglia, *Expressing a random variable in terms of uniform random variables*, Annals Math. Stat. 32 (1961), pp. 894–898.
6. G. Marsaglia, *Generating exponential random variables*, Annals Math. Stat. 32 (1961), pp. 899–900.
7. G. Marsaglia, *Generating a variable from the tail of the normal distribution*, Technometrics Vol. 6, no. 1, February 1964, pp. 101–102.
8. G. Marsaglia, M. D. MacLaren, T. A. Bray, *A fast procedure for generating normal random variables*, Comm. ACM, Vol. 7, no. 1, Jan. 1964, pp. 4–10.
9. M. D. MacLaren, G. Marsaglia, T. A. Bray, *A fast procedure for generating exponential random variables*, Comm. ACM, Vol. 7, no. 5, May 1964, pp. 298–300.
10. M. E. Muller, *A comparison of methods for generating normal deviates on digital computers*, J. Assoc. Comp. Mach. 6 (1959), pp. 376–383.
11. Masaki Sibuya, *On exponential and other random variable generators*, Annals of the Institute of Stat. Math., Tokyo, Vol. 13 (1961/62): 3, pp. 231–237.
12. S. S. Gupta, *Probability integrals of multivariate normal and multivariate t*, Annals Math. Stat. 34 (1963), pp. 792–828.
13. S. S. Gupta, *Bibliography on the multivariate normal integrals and related topics*, Annals Math. Stat. 34 (1963), pp. 829–838.

RESEARCH INSTITUTE OF NATIONAL DEFENCE,
STOCKHOLM, SWEDEN