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The probability generating function of the Freund-Ansari-Bradley statistic

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Abstract

We derive a expression for the probability generating function of the distribution-free Freund-Ansari-Bradley scale statistic. From this generating function we show how to systematically compute the exact null distribution and the moments of the statistic within the computer algebra system Mathematica. Finally, we give a table with critical values which extends the existing tables.

Keywords Freund-Ansari-Bradley test, generating functions, exact distribution, moments, computer algebra, nonparametric statistics.

1 Introduction

In this article we derive a expression for the probability generating function of the distribution-free Freund-Ansari-Bradley statistic, which is abbreviated as the FAB statistic. Statistically equivalent versions of this statistic were introduced by Freund and Ansari (1957) and Ansari and Bradley (1960). We implemented this generating function in the computer algebra system Mathematica for computing the distribution of the test statistic very quickly. We also show how to use the generating function for computing (higher) moments of the test statistic. In the first appendix we present the Mathematica code for expanding the generating function, in the second appendix we give a table with critical values for the balanced cases which extends the existing tables from $N \leq 20$ to $N \leq 80$. For general details about the FAB statistic we refer to Gibbons and Chakraborti (1992).

Several methods have been developed for computing the null distribution of the FAB statistic. Ansari and Bradley (1960) and Kannemann (1983) derive recurrence relations based on a two-dimensional generating function of Euler (1748, transl. 1988). Other methods are general methods for computing null distributions of linear two-sample rank tests and hence do not use the specific characteristics of the FAB statistic. Examples of these methods include the Pagano and Tritchler (1983) approach based on characteristic functions and fast Fourier transforms and the network algorithm developed by Mehta et al. (1987). For a short description of these methods we refer to Good (1994). We give an alternative method based on the two-dimensional generating function in Van de Wiel (1996). This last method is fast and, although not trivial, very intuitive. We use exact expressions and do not deal with rounding errors as some recursive methods do.

2 The Freund-Ansari-Bradley test

Let $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ be independent samples from continuous distribution functions. Thus we may and will assume that ties do not occur. We consider the combined sample $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$, $N = m+n$. The Freund-Ansari-Bradley test is a two-sample scale test. The corresponding test statistic is defined by:

$$A_N = \sum_{\ell=1}^{N} \left| \left(\ell - \frac{N+1}{2} \right) \right| Z_\ell, \tag{1}$$

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where $Z_\ell = 1$ if the ℓth order statistic in the combined sample is an X-observation and $Z_\ell = 0$ otherwise. Thus, for this statistic the rank scores are

$$a(\ell) = \left| \left(\ell - \frac{N+1}{2} \right) \right| , \tag{2}$$

$\ell = 1, \ldots, N$.

Duran (1976) compares the FAB statistic with other scale statistics like the Mood scale statistic, the Siegel-Tukey statistic and the Klotz statistic (for details about these test statistics: see Gibbons and Chakraborti (1992)). It turns out that the Siegel-Tukey statistic and FAB statistic are asymptotically equivalent in terms of (Pitman) asymptotic relative efficiency. The Mood statistic and Klotz statistic are more efficient when the alternative is normal (or light-tailed), while the Siegel-Tukey statistic and FAB statistic are more efficient for heavy-tailed distributions. The FAB statistic attains much less values than the Mood and Klotz statistic. This is both a computational advantage and a potential disadvantage regarding the number of significance levels (for small sample sizes).

3 The probability generating function

From (2) we see that for N even the rank scores are of the form $i - \frac{1}{2}, i = 1, \ldots \frac{N}{2}$. We will find it to be convenient that the scores are integers and therefore we introduce adjusted FAB scores:

$$a'(\ell) = \begin{cases} \left| \left(\ell - \frac{N+1}{2} \right) \right| + \frac{1}{2} & \text{if } N \text{ is even} \\ \left| \left(\ell - \frac{N+1}{2} \right) \right| & \text{if } N \text{ is odd,} \end{cases} \tag{3}$$

$\ell = 1, \ldots, N$. We define the adjusted FAB test statistic A'_N as A_N with the FAB scores replaced by the adjusted FAB scores. The statistic A'_N is, of course, statistically equivalent to A_N. For N even, we may split the set of adjusted scores $\{a'(1), \ldots, a'(N)\}$ into two sets of Wilcoxon scores $\{1, \ldots, \frac{1}{2}N\}$, which is the main idea behind our approach. With the aid of this observation we found the following theorem for the case N even:

Theorem 3.1 *Under H_0, $N = m + n$ even and $m \leq n$ the probability generating function of the Freund-Ansari-Bradley statistic is*

$$\sum_{k=0}^{\infty} \Pr(A_N = k)\, x^k = \frac{1}{q^{\frac{m}{2}} \binom{N}{m}} \sum_{i=0}^{m} c(i)\, c(m-i) \begin{bmatrix} \frac{N}{2} \\ i \end{bmatrix}_q \begin{bmatrix} \frac{N}{2} \\ m-i \end{bmatrix}_q , \tag{4}$$

where $\begin{bmatrix} r \\ 0 \end{bmatrix}_q = 1$, $\begin{bmatrix} r \\ s \end{bmatrix}_q = \frac{\prod_{\ell=1}^{r} (1-q^\ell)}{\prod_{\ell=1}^{s} (1-q^\ell) \prod_{\ell=1}^{r-s} (1-q^\ell)}$ *for* $s > 0$ *and* $c(i) = q^{\frac{i}{2}(i+1)}$.

Proof: Assume, without loss of generalisation, that $m \leq n$. We define R_i as the rank score in the pooled sample corresponding to $X_i, i = 1, \ldots, m$. Similarly, S_j is the rank score corresponding to $Y_j, j = 1, \ldots, n$. We know that under H_0 every configuration r of $R_1, \ldots, R_m, S_1, \ldots, S_n$ is equiprobable. Let $T(a)$ be the first half of a configuration a. We can partition these configurations into classes of configurations with i R's in $T(a)$, $i \leq m$. The class of configurations with i R's in $T(a)$ is called C_i. We denote the elements of $T(a)$ by $R_{u_1}, \ldots R_{u_i}, S_{v_1}, \ldots, S_{v_{N/2-i}}$, where $u_1, \ldots, u_i$ and $v_1, \ldots, v_{N/2-i}$ are subsequences of $1, \ldots, m$ and $1, \ldots, n$, respectively. We define the following Wilcoxon statistics:

$$W_{i,N/2-i} = \sum_{j=1}^{i} R_{u_j} \quad \text{and} \quad W_{m-i,N/2-m+i} = \sum_{j=1}^{m-i} R_{u_{i+j}}.$$

We need to prove the conditional independence of $W_{i,N/2-i}$ and $W_{m-i,N/2-m+i}$, given C_i. Let $K_1(a)$ be the class of r's for which the first half equals the first half of a configuration a. Similarly, let $K_2(a)$ be the class of r's for which the second half equals the second half of a. We note that the event $r = a$ is equivalent to $r \in K_1(a) \cap r \in K_2(a)$. There are $\binom{N/2}{i}\binom{N/2}{m-i}$ configurations in C_i. They are equiprobable under H_0, so

$$\Pr(r = a | r \in C_i) = \begin{cases} \frac{1}{\binom{N/2}{i}\binom{N/2}{m-i}} & \text{if } a \in C_i \\ 0 & \text{if } a \notin C_i. \end{cases} \tag{5}$$

Under H_0 we have,

$$\begin{aligned} \Pr(r \in K_1(a) | r \in C_i) &= \begin{cases} \frac{\#(r: r \in K_1(a) \cap r \in C_i)}{\#(r: r \in C_i)} = \frac{\binom{N/2}{m-i}}{\binom{N/2}{i}\binom{N/2}{m-i}} = \frac{1}{\binom{N/2}{i}} & \text{if } a \in C_i \\ 0 & \text{if } a \notin C_i, \end{cases} \\ \Pr(r \in K_2(a) | r \in C_i) &= \begin{cases} \frac{1}{\binom{N/2}{m-i}} & \text{if } a \in C_i \\ 0 & \text{if } a \notin C_i. \end{cases} \end{aligned} \tag{6}$$

Combining (5) and (6) we conclude that

$$\begin{aligned} \Pr(r \in K_1(a) \cap r \in K_2(a) | r \in C_i) &= \Pr(r = a | r \in C_i) \\ &= \Pr(r \in K_1(a) | r \in C_i) \Pr(r \in K_2(a) | r \in C_i). \end{aligned} \tag{7}$$

Let r_i^1 be a configuration of i R's and $N/2 - i$ S's and let r_i^2 be a configuration of $m - i$ R's and $N/2 - m + i$ S's. For $j = 1, 2$:

$$\Pr(r \in K_j(a) | r \in C_i) = \widetilde{\Pr}(r_i^j = a_j), \tag{8}$$

where a_j is the jth ($j = 1, 2$) half of the configuration a. The symbol Pr denotes the probability measure on the space with configurations of length N, whereas $\widetilde{\Pr}$ denotes the probability measure on the space with configurations of length $N/2$. The statistics $W_{i,N/2-i}$ and $W_{m-i,N/2-m+i}$ are functions of r_i^1 and r_i^2, respectively. Because of equality (8) we may regard $W_{i,N/2-i}$ and $W_{m-i,N/2-m+i}$ as functions of all r's for which $r \in C_i$. Since equation (7) tells us that, given $r \in C_i$, the events $\{r \in K_1(a)\}$ and $\{r \in K_2(a)\}$ are independent, we conclude that $W_{i,N/2-i}$ and $W_{m-i,N/2-m+i}$ are also independent, given C_i.

The set of adjusted FAB scores consists of two identical sets of Wilcoxon scores. When i scores of the first set are assigned to the X's, we know that $m - i$ scores of the second set are assigned to the X's, $0 \le i \le m$. The sum of the scores assigned to X's equals A'_N and it also equals the sum of $W_{i,N/2-i}$ and $W_{m-i,N/2-m+i}$. Therefore, $\#(A'_N = k) = \sum_{i=0}^{m} \#(W_{i,N/2-i} + W_{m-i,N/2-m+i} = k)$. Let H_Z be a generating function for the number of ways a statistic Z can reach a certain value. Then,

$$\begin{aligned} H_{A'_N} &= \sum_{k=0}^{\infty} \#(A'_N = k)\, x^k = \sum_{k=0}^{\infty} \sum_{i=0}^{m} \#(W_{i,\frac{N}{2}-i} + W_{m-i,\frac{N}{2}-m+i} = k)\, x^k \\ &= \sum_{i=0}^{m} \sum_{k=0}^{\infty} \#(W_{i,\frac{N}{2}-i} + W_{m-i,\frac{N}{2}-m+i} = k)\, x^k = \sum_{i=0}^{m} H_{W_{i,\frac{N}{2}-i} + W_{m-i,\frac{N}{2}-m+i}} \\ &= \sum_{i=0}^{m} H_{W_{i,\frac{N}{2}-i}} H_{W_{m-i,\frac{N}{2}-m+i}}, \end{aligned} \tag{9}$$

where in the last step we used that $W_{i,\frac{N}{2}-i}$ and $W_{m-i,\frac{N}{2}-m+i}$ are independent. The generating function $H_{W_{a,b}}$ can easily be derived from the generating function of the equivalent Mann-Whitney statistic $M_{a,b}$. We know that $W_{a,b} = M_{a,b} + \frac{1}{2}a(a+1)$ and from Andrews (1976, Ch. 3) and David and Barton (1962, pp. 203-204) we know that

$$H_{M_{a,b}} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q .$$

Therefore,

$$H_{W_{a,b}} = c(a) \begin{bmatrix} a+b \\ a \end{bmatrix}_q . \qquad (10)$$

Substituting (10) into (9) with $(a,b) = (i, \frac{N}{2} - i)$ and $(a,b) = (m-i, \frac{N}{2} - m + i)$ gives us $H_{A'_N}$. We complete our proof by remarking that for N even, $A_N = A'_N - \frac{m}{2}$. □

Theorem 3.2 *Under H_0, $N = m + n$ odd and $m \leq n$ the probability generating function of the Freund-Ansari-Bradley statistic is*

$$\sum_{k=0}^{\infty} \Pr(A_N = k)\, x^k = \frac{1}{\binom{N}{m}} \sum_{j=0}^{1} \sum_{i=0}^{m-j} c(i)\, c(m-j-i) \begin{bmatrix} \frac{N-1}{2} \\ i \end{bmatrix}_q \begin{bmatrix} \frac{N-1}{2} \\ m-j-i \end{bmatrix}_q \qquad (11)$$

where $\begin{bmatrix} r \\ s \end{bmatrix}_q$ *and $c(i)$ as in Theorem 3.1.*

Proof: The proof is similar to that of Theorem 3.1, but now we deal with two identical sets of Wilcoxon scores and one score that is equal to zero. This problem is solved by summing over a variable j that equals 1 if zero is assigned to an X-observation and that equals 0 otherwise. So for $j = 0$ we obtain all possible values of A'_N with m scores in the two identical sets and for $j = 1$ we obtain all possible values of A'_N with $m - 1$ scores in the two identical sets. □

4 Moments of the FAB statistic

Formulas (4) and (11) tell us that for computing moments of the FAB statistic it suffices to compute derivatives of the expression

$$V(q) = d(q) \begin{bmatrix} r \\ s \end{bmatrix}_q \begin{bmatrix} r \\ t \end{bmatrix}_q ,$$

where $d(q)$ is a finite sum of terms of the form $c\, q^z$, where c and z are rational. If we denote the kth derivative of $f(q)$ by $f^{(k)}(q)$ then we see that

$$V^{(k)}(q) = \sum_{j=0}^{k-i} \sum_{i=0}^{k} \binom{k}{k-i-j, i, j} d^{(k-i-j)}(q) \begin{bmatrix} r \\ s \end{bmatrix}_q^{(i)} \begin{bmatrix} r \\ t \end{bmatrix}_q^{(j)} \qquad (12)$$

Since $d(q)$ is a finite sum of terms of the form $c\, q^z$, where c and z are rational, it is straightforward to compute $d^{(\ell)}(1)$ for ℓ arbitrary large. The two other terms in the right hand side of (12) are polynomials, so we may take the limit $q \to 1$ and we find the following expression for the kth derivative of $V(q)$ at $q = 1$:

$$V^{(k)}(1) = \sum_{j=0}^{k-i} \sum_{i=0}^{k} \binom{k}{k-i-j, i, j} d^{(k-i-j)}(1) \lim_{q \to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q^{(i)} \lim_{q \to 1} \begin{bmatrix} r \\ t \end{bmatrix}_q^{(j)} \qquad (13)$$

Di Bucchianico (1996) provides a method for computing expressions of the form

$$\lim_{q \to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q^{(i)}$$

with the aid of the computer algebra package Mathematica. We used his method for computing moments of the FAB statistic.

4.1 Example

As an illustration we compute the mean for the case N even. In this case $k = 1$, so after expanding (13) we get

$$V'(1) = d(1) \lim_{q\to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q' \lim_{q\to 1} \begin{bmatrix} r \\ t \end{bmatrix}_q + d(1) \lim_{q\to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q \lim_{q\to 1} \begin{bmatrix} r \\ t \end{bmatrix}_q' + d'(1) \lim_{q\to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q \lim_{q\to 1} \begin{bmatrix} r \\ t \end{bmatrix}_q \tag{14}$$

In this case $d(q) = \binom{N}{m}^{-1} q^{\frac{1}{2}i(i+1)+\frac{1}{2}(m-i)(m-i+1)-\frac{m}{2}}$, so $d(1) = \binom{N}{m}^{-1}$ and $d'(1) = \binom{N}{m}^{-1}(\frac{1}{2}i(i+1) + \frac{1}{2}(m-i)(m-i+1) - \frac{m}{2}) = \frac{1}{2}\binom{N}{m}^{-1}(2i^2 + m^2 - 2im)$. From the example in Di Bucchianico (1996,p. 9) we extract that

$$\lim_{q\to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q = \binom{r}{s} \text{ and } \lim_{q\to 1} \begin{bmatrix} r \\ s \end{bmatrix}_q' = \frac{1}{2}\binom{r}{s}(r-s)s.$$

The same equations hold for s replaced by t. In equation (4) we see that $r = \frac{N}{2}, s = i$ and $t = m - i$. Substituting in (14) leaves us after some labour

$$V'(1) = \frac{mN\binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{m-i}}{4\binom{N}{m}} \tag{15}$$

The last thing we have to do is summing over i and we get the mean μ_{A_N} for N even:

$$\mu_{A_N} = \sum_{i=0}^{m} \frac{mN\binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{m-i}}{4\binom{N}{m}} = \frac{mN}{4}, \tag{16}$$

where we use the fact that $\sum_{i=0}^{m} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{m-i} = \binom{N}{m}$ which is a special case of the Chu-VanderMonde formula.[2] For a complete proof: see (Chu 1303, transl. 1959) or Riordan (1979). .

4.2 Example of a higher moment

We give $E\,A_N^5$, the fifth moment of the FAB statistic for N even. It took 80 seconds to compute this moment on a SunSparc 10.

$$E\,A_N^5 = \frac{m\,P(m,n)}{3072\,(m+n-3)\,(m+n-1)} \quad \text{with}$$

$P(m,n) =$
$3\,m^{11} + m^{10}\,(-12 + 21\,n) + m^9\,(-111 - 62\,n + 63\,n^2) +$
$m^8\,(480 - 705\,n - 120\,n^2 + 105\,n^3) + m^7\,(1320 + 2118\,n - 1855\,n^2 - 90\,n^3 + 105\,n^4) +$
$m^6\,(-6720 + 7804\,n + 3420\,n^2 - 2585\,n^3 + 20\,n^4 + 63\,n^5) +$
$m^5\,(-4560 - 24216\,n + 17804\,n^2 + 2098\,n^3 - 2005\,n^4 + 78\,n^5 + 21\,n^6) +$
$m^4\,(38400 - 31296\,n - 30416\,n^2 + 19752\,n^3 - 248\,n^4 - 811\,n^5 + 48\,n^6 + 3\,n^7) +$
$m^3\,(-10368 + 105120\,n - 62896\,n^2 - 13264\,n^3 + 10672\,n^4 - 822\,n^5 - 125\,n^6 + 10\,n^7) +$
$m^2\,(-73728 + 27968\,n + 90752\,n^2 - 50512\,n^3 + 1472\,n^4 + 2204\,n^5 - 260\,n^6 + 5\,n^7) +$
$m\,(55296 - 133888\,n + 64192\,n^2 + 20768\,n^3 - 14384\,n^4 + 1832\,n^5 - 36\,n^6 - 2\,n^7) +$
$16896\,n - 60160\,n^2 + 28544\,n^3 - 3264\,n^4 - 32\,n^5 + 16\,n^6$

[2] Sketch of the proof: define a generating function with coefficients equal to the left-hand side of the equality and with summing variable m. Then use the convolution theorem to split the sum into a product of two sums. Finally, use the binomial theorem to obtain the desired result.

5 Computer algebra

This section contains the text of the Mathematica package we used for computing the distribution of the FAB statistic by using Theorems 3.1 and 3.2. We also give a small example.

```
FABevengf[N_,m_]:= Expand[Simplify[1/Binomial[N,m]*q^(-(m/2))*
(Sum[c[i]*c[m-i]*Product[1-q^l,{l,N/2}]/(Product[1-q^l,{l,i}]*
Product[1-q^l,{l,N/2-i}])*Product[1-q^l,{l,N/2}]/
(Product[1-q^l,{l,m-i}]*Product[1-q^l,{l,N/2-(m-i)}]),{i,m-1}] +
(2*c[m])*Product[1-q^l,{l,N/2}]/
(Product[1-q^l,{l,m}]*Product[1-q^l,{l,N/2-m}]))]]

FABoddgf[N_,m_]:= Expand[Simplify[1/Binomial[N,m]*
Sum[Sum[c[i]*c[m-j-i]*Product[1-q^l,{l,(N-1)/2}]/
(Product[1-q^l,{l,i}]*Product[1-q^l,{l,(N-1)/2-i}])*
Product[1-q^l,{l,(N-1)/2}]/(Product[1-q^l,{l,m-j-i}]*
Product[1-q^l,{l,(N-1)/2-(m-j-i)}]),{i,m-j-1}] +
(2*c[m-j])*Product[1-q^l,{l,(N-1)/2}]/(Product[1-q^l,{l,m-j}]*
Product[1-q^l,{l,(N-1)/2-(m-j)}]),{j,0,1}]]]

c[i_]:= q^((1/2)*i*(i+1))
```

Note that the cases $i = 0$ and $i = m-j$ are split off, because we have to tell Mathematica explicitly that $\begin{bmatrix} r \\ 0 \end{bmatrix}_q = 1$.

The contribution of these cases to the sum is equal. The Mathematica functions `Expand` and `Simplify` are used to compute the full polynomial which represents the distribution of the FAB statistic. The following example gives the distribution of the FAB statistic for $m = n = 4$.

```
FABevengf[8,4]

 4      5      6      7      8      9      10      11     12
q    2 q    9 q    6 q    9 q    6 q    9 q     2 q     q
-- + ---- + ---- + ---- + ---- + ---- + ----- + ----- + ---
70    35     70     35     35     35     70      35     70
```

6 Table of critical values

With the aid of Theorems 3.1 and 3.2 we were able to extend the existing tables of critical values. Ansari and Bradley (1960) give critical values for $N \leq 20$. We give tables for $n = m, N \leq 80$. For practical reasons we did not print the unbalanced cases. Anyone interested in critical values for an unbalanced case may contact the author.

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n	0.005	0.01	0.025	0.05	0.1	0.1	0.05	0.025	0.01	0.005
4	*	*	4	4	5	11	12	12	*	*
5	*	6.5	7.5	7.5	8.5	16.5	17.5	17.5	18.5	*
6	9	10	11	12	13	23	24	25	26	27
7	13.5	14.5	15.5	17.5	18.5	30.5	31.5	33.5	34.5	35.5
8	19	20	22	23	25	39	41	42	44	45
9	25.5	26.5	28.5	30.5	32.5	48.5	50.5	52.5	54.5	55.5
10	33	34	36	38	40	60	62	64	66	67
11	40.5	42.5	44.5	46.5	49.5	71.5	74.5	76.5	78.5	80.5
12	49	51	54	57	60	84	87	90	93	95
13	59.5	61.5	64.5	67.5	70.5	98.5	101.5	104.5	107.5	109.5
14	70	72	76	79	83	113	117	120	124	126
15	81.5	83.5	88.5	91.5	95.5	129.5	133.5	136.5	141.5	143.5
16	93	97	101	105	110	146	151	155	159	163
17	106.5	110.5	115.5	119.5	124.5	164.5	169.5	173.5	178.5	182.5
18	121	125	130	135	141	183	189	194	199	203
19	136.5	140.5	146.5	151.5	157.5	203.5	209.5	214.5	220.5	224.5
20	152	156	163	169	175	225	231	237	244	248
21	169.5	173.5	180.5	186.5	193.5	247.5	254.5	260.5	267.5	271.5
22	187	192	199	206	214	270	278	285	292	297
23	205.5	211.5	219.5	226.5	234.5	294.5	302.5	309.5	317.5	323.5
24	225	231	240	247	256	320	329	336	345	351
25	245.5	252.5	261.5	269.5	278.5	346.5	355.5	363.5	372.5	379.5
26	267	274	284	292	302	374	384	392	402	409
27	289.5	296.5	307.5	315.5	326.5	402.5	413.5	421.5	432.5	439.5
28	313	321	331	341	352	432	443	453	463	471
29	337.5	345.5	356.5	366.5	378.5	462.5	474.5	484.5	495.5	503.5
30	363	371	383	393	406	494	507	517	529	537
31	388.5	397.5	410.5	421.5	433.5	527.5	539.5	550.5	563.5	572.5
32	416	425	438	450	463	561	574	586	599	608
33	443.5	453.5	467.5	479.5	493.5	595.5	609.5	621.5	635.5	645.5
34	473	483	497	510	525	631	646	659	673	683
35	502.5	513.5	528.5	541.5	556.5	668.5	683.5	696.5	711.5	722.5
36	534	544	560	574	590	706	722	736	752	762
37	565.5	576.5	593.5	607.5	624.5	744.5	761.5	775.5	792.5	803.5
38	598	610	627	642	659	785	802	817	834	846
39	631.5	643.5	661.5	677.5	695.5	825.5	843.5	859.5	877.5	889.5
40	666	679	697	714	732	868	886	903	921	934

Table 1: Left and right critical values for the Freund-Ansari-Bradley test, $n = m$.

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